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I^{*} -ALGEBRAS AND THEIR APPLICATIONS

A Thesis

presented for the Degree of Doctor of Philosophy
in Mathematics in the Open University

by

Julio César Alcántara-Bode, B.S., M.Sc.

July 1979

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CORRECTIONS TO THESIS "I*-ALGEBRAS AND THEIR APPLICATIONS".

The proof in Proposition 2.3 of the fact that \bar{A}_+ is proper, i.e., $\bar{A}_+ \cap -\bar{A}_+ = \langle 0 \rangle$, is not correct. This only affects the contents of Chapter 2. If we define a "proper I*-algebra" as an I*-algebra for which \bar{A}_+ is proper then all the results of Chapter 2 remain true for proper I*-algebras.

G.Lassner and A.Uhlmann [C.M.P.7(1968) 152-159] show that $\underline{S}(\mathbb{R}^n)$ and $\underline{D}(\mathbb{R}^n)$ are proper I*-algebras. Their method easily extends to ${}^2\underline{D}(\mathbb{R}^n)$ and $\underline{D}(\mathbb{R}^n, \mathcal{G})$. I do not know of an example of an I*-algebra that is not proper.

On page 43 lines -11,-12, replace " $P\pi_{\phi}^{**}(\mathcal{A})P$ " by "the subalgebra generated by $P\pi_{\phi}^{**}(\mathcal{A})P$."

It is worth pointing out that the algebraic axioms in Definition 1.1, no divisors of zero, trivial centre and invertibility only for the non-zero elements of the centre, are only used in Propositions 1.4, 1.5, Corollary 1.6 and part (c) of Proposition 2.10. For this reason they have been dropped in the definition of I*-algebras in the joint paper with D.A. Dubin, I*-Algebras and their Applications (submitted to C.M.P.). They then follow as special properties of certain I*-algebras. In particular, they are true for all BU-algebras.

ABSTRACT OF THESIS

I^* -ALGEBRAS AND THEIR APPLICATIONS

Presented for the Degree of Doctor of Philosophy

in the Open University

by

Julio César Alcántara-Bode, B.S., M.Sc.

July 1979

In this thesis we study a class of topological I^* -algebras, I^* -algebras, that subsume and generalize the Borchers algebra $\mathcal{L}(\mathbb{R}^4)$.

In Chapter 1 we define an I^* -algebra as a complex unital topological I^* -algebra with no zero divisors, trivial centre, whose only invertible elements are the non-zero elements of the centre; and such that if $\sum_{i=1}^n f_i^* f_i = 0$, then $f_i = 0$ ($1 \leq i \leq n$). As a locally convex space it is a nuclear LF-space. We derive some algebraic and topological properties from these axioms and show that completed tensor algebras over nuclear LF-spaces (BU-algebras) are I^* -algebras.

In Chapter 2 we isolate a certain subclass of I^* -algebras of which $\mathcal{L}(\mathbb{R}^n)$ is typical, by defining what we call property (N). Such algebras will be seen to possess an extensive list of topological and order properties.

In Chapter 3 we introduce BU-algebras convenient for the study of the CAR, CCR and Current Commutation Relations. We show that for the $su(2)$ Current Algebra, the formal Heisenberg Hamiltonian admits a rigorous interpretation as a derivation on the corresponding BU-algebra. We characterize states in which this derivation is implemented. A procedure is given to construct representations of Current Algebra in terms of Fréchet-Volterra derivatives. For the $su(2)$ Current Algebra we give an example of a representation with a prime part: the corresponding state is not infinitely divisible, therefore.

In Chapter 4 we discuss symmetries. Under some quite general hypotheses we are able to reproduce most of the results known in the C^* -algebraic framework.

In Chapter 5 we show that the geometrical approach to the Tomita-

Takesaki theory, due to Rieffel and Van Daele, is suitable for extension to general topological $*$ -algebras. This enables us to introduce the notion of KMS states on topological $*$ -algebras, and to show that Bogolubov's and Sewell's inequalities hold for such states. With an additional technical assumption of operator domain stability, Sewell's inequality will be shown to be equivalent to the KMS condition.

In Appendices A, B, and C we summarize the definitions and the results from the theory of locally convex spaces, ordered vector spaces, and semigroups in locally convex spaces, respectively, that are employed in the main body of the thesis.

In Appendix D we illustrate the ideas of Rieffel and Van Daele, constructing a family of modular automorphisms, independently of Von Neumann algebra considerations.

Originality

It is worth pointing out at the outset that the principal results in Chapters 1 and 2 are patterned after the known properties of the Borchers algebra $\mathcal{F}(\mathbb{R}^4)$. We have mostly given the proofs that are rather different and more general than those for $\mathcal{F}(\mathbb{R}^4)$. For Chapters 4 and 5, the ergodic and dynamical properties of C^* -algebras served as a prototype.

In Chapter 1 originality is claimed for the order axiom and the LF property in the topological axiom of Definition 1.1, Proposition 1.5, Corollary 1.6, Definition 1.7, Theorem 1.8, Examples 1.9(b) and (c), the second isomorphism in Proposition 1.10, the proof of Lemma 1.11, Proposition 1.12 and Corollary 1.13.

From the results given in Chapter 2 we claim originality for the part of Proposition 2.4 that does not involve the nuclearity of \mathcal{M} , Proposition 2.5, the isolation of Property (N), Proposition 2.6, the last part of Proposition 2.8, Proposition 2.9 and part (c) of Proposition 2.10.

As far as the results of Chapter 3 are concerned, we claim originality for Proposition 3.11 except its last part, the procedure to construct representations of Current Algebra in terms of Fréchet-Volterra derivatives, and the example of a representation with a prime part.

In Chapter 4 originality is claimed for Proposition 4.6, Lemma 4.7, Proposition 4.8, part of the proof of Theorem 4.9, and Proposition 4.11.

In Chapter 5 the application of the Rieffel-Van Daele theory to general topological $*$ -algebras, even though quite straightforward, is new, as is the derivation of Sewell's inequality (Proposition 5.12). Proposition 5.7 is new in the stated generality.

For the material in the Appendices our only claim to originality is to Propositions A.49, A.50, and the whole of Appendix D.

ACKNOWLEDGEMENTS

I would like to thank my supervisor, Dr. D.A. Dubin, for focusing my attention on the problems that led to this research, discussions, suggestions, and constructive criticism; and Professor S. Ōuchi for sending me a proof of Proposition C.9. I am also grateful to the following institutions: the British Council, for financial support during the period 1976-79; U.P.C.H. (LIMA-PERU) for an extended leave of absence during the same period; the Mathematical Institute, Oxford, for permission to use their research facilities; and last but by no means least the Faculty of Mathematics of the Open University. In particular I would like to mention Ellison Platt for his kind hospitality; and Mrs. Vivienne Yarwood for her efficient and competent typing of what was a very complicated manuscript.

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CHAPTER 1

I^* -ALGEBRAS

Following the work of Borchers [1] and Uhlmann [2], the Wightman Quantum Field Theory of a scalar field may be formulated in terms of representations of a certain topological I^* -algebra. In this way one obtains a very concise description of Quantum Field Theory, and, more importantly, a suitable framework within which one can study some important and general structural questions [3-7].

Under some additional technical hypotheses, this formulation of Quantum Field Theory reduces to a C^* -algebraic one [8,9]; unfortunately, the physical meaning of these extra hypotheses is unclear.

1.1 Definition and Immediate Consequences

It is our objective in this section to single out a class of topological I^* -algebras relevant to the study of Current Algebras, quantum continuous systems in statistical mechanics, and complex and/or multicomponents fields in Quantum Field Theory. The class of algebras so defined we shall call I^* -algebras; the axioms that follow are motivated by properties of the Borchers algebra $\mathcal{P}(\mathbb{R}^4)$ [10,11].

Definition 1.1 An I^* -algebra is a complex unital topological I^* -algebra with no zero divisors and trivial centre. Moreover the only invertible elements are the non-zero elements of the centre; and if $\sum_{i=1}^n f_i^* f_i = 0$, then $f_i = 0$ ($1 \leq i \leq n$). As a locally convex space it is a nuclear LF-space.

Note that for a topological I^* -algebra, the product is separately, continuous and the involution is continuous.

In the next Proposition we collect some topological properties of I^* -algebras needed for further developments.

Proposition 1.2 An I^* -algebra A is barrelled, bornological, complete Montel and reflexive. Its strong dual, A' , is nuclear, complete, barrelled, bornological, and Montel.

Proof As A is an LF-space, it is barrelled, bornological, and complete : [12], Cor 2 (p61); Cor 2 (p62); and Cor (p60). Being complete, barrelled, and nuclear, implies that A is a Montel space : [13], Cor 3 (p520); consequently it is reflexive : [12], p 147.

As A is nuclear LF, A' is nuclear : [12], Theorem 9.6 (p172); exercise 2 (p173). A' is complete because A is bornological : [12], Theorem 6.1 (p148). As A is nuclear, it is a Schwartz space ([14]); A' is then the strong dual of a complete Schwartz space and consequently barrelled and bornological : [15], prob 9 (p287). Finally, A' is Montel because A is : [12], Theorem 5.9 (p147). ■

In the discussion of the order properties of I^* -algebras we will need the following improvement on the continuity properties of the product.

Proposition 1.3 The product in an I^* -algebra A is hypocontinuous, therefore jointly continuous on bounded sets. The product of bounded sets is bounded.

Proof Since A is barrelled, by Proposition 1.2, and the product is separately, continuous, its hypocontinuity follows from [15], Theorem 2 (p360). The other two assertions follow directly from the hypocontinuity of the product : [15], Proposition 2 (p359). ■

For the sake of completeness we mention the following algebraic properties of I^* -algebras.

Proposition 1.4 In an I^* -algebra, the only idempotents are zero and the identity. There are no proper minimal ideals, and the Jacobson radical is $\{0\}$.

Proof See [10], Theorem II.5; Theorem II.8; and Theorem II.6, respectively. ■

The next proposition shows that the only interesting I^* -algebras are infinite dimensional.

Proposition 1.5 If an I^* -algebra A is different from its centre, it is infinite dimensional.

Proof Suppose A is finite dimensional. It must then be isomorphic to a Banach algebra with identity and as such is an algebra with continuous inverse : [16], p177. Then for $\lambda \in \mathbb{C}$, $|\lambda|$ sufficiently large, $f - \lambda$ is invertible, so $f - \lambda = \mu$, for some non-zero $\mu \in \mathbb{C}$. Therefore the algebra is equal to its centre. ■

Corollary 1.6 The only finite dimensional I^* -algebra is \mathbb{C} .

1.2 BU-Algebras

The question of the existence of non-trivial I^* -algebras is answered affirmatively by the following important example of a Borchers-Uhlmann algebra, or BU-algebra. Indeed, we do not know of any I^* -algebras, apart from \mathbb{C} , which are not of this form.

Definition 1.7 Let $E_{\mathbb{R}}$ be a real nuclear LF-space and $E = E_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ its complexification. The BU-algebra over E is the locally convex direct sum

$$\underline{E} = \bigoplus_{n=0}^{\infty} \overline{\otimes}^n E \quad (1.1)$$

where $n=0$ corresponds to \mathbb{C} by convention, and $\overline{\otimes}$ indicates the completion of the tensor product in the inductive tensor product topology (see Appendix A). The product with respect to which \underline{E} is an algebra follows from its graded structure: if $\underline{f} = (f_0, f_1, \dots, f_r, 0, 0, \dots)$, $\underline{g} = (g_0, g_1, \dots, g_s, 0, 0, \dots) \in E$, then

$$\underline{f} \times \underline{g} = (f_0 g_0, f_0 g_1 + f_1 g_0, \dots, \sum_{i+j=p} f_i \otimes g_j, \dots, f_r \otimes g_s, 0, 0, \dots) \quad (1.2)$$

It is further assumed that a continuous involution, J , is defined on $E_{\mathbb{R}}$. In an obvious way this extends linearly to an involution

$$\underline{f} \mapsto \underline{f}^* \text{ on } E, \text{ with } (\lambda \underline{f} \times \underline{g})^* = \overline{\lambda} \underline{g}^* \times \underline{f}^*$$

Theorem 1.8 A BU-algebra is an I^* -algebra.

Proof For the algebraic properties we may modify Lemma I.2.4 of [11] slightly. The identity is $\underline{1} = (1, 0, 0, \dots)$. For the divisors of zero we come to $\underline{f}_r \otimes \underline{g}_s = 0$ with $\underline{f}_r \in \overline{\otimes}^r E$ and $\underline{g}_s \in \overline{\otimes}^s E$, $\underline{g}_s \neq 0$ (c.f. [11] ibid). Then $\underline{f}_r = 0$ by the linear disjointness of tensor products : [13], p.403. If \underline{f} is in the centre of the algebra, choosing elements \underline{g} such that $g_j = 0$ unless $j = r$, where $f_r \neq 0$, $f_i = 0$ for all $i > r$, then $\underline{f}_r \otimes \underline{g}_r = \underline{g}_r \otimes \underline{f}_r$ for all $\underline{g}_r \in \overline{\otimes}^r E$. The case $r=0$ gives $\underline{f} \in \mathbb{C} \underline{1}$, so assume $r > 0$. By linear disjointness it follows that $\underline{f}_r = \lambda \underline{g}_r$ ($\lambda \in \mathbb{C}$) for all $\underline{g}_r \in \overline{\otimes}^r E$, implying $\underline{f}_r = 0$ which is a contradiction. This proves the algebraic properties.

Since E is a nuclear LF-space, $\overline{\otimes}^n E$, $n \geq 2$, and \underline{E} are nuclear LF-

spaces by Propositions A.49 and A.50 of Appendix A, respectively.

We prove now the separate continuity of the product: the map

$L(\underline{f}) : \underline{E} \longrightarrow \underline{E}$, $L(\underline{f})(\underline{g}) = \underline{f} \times \underline{g}$ is continuous iff the restriction of

$L(\underline{f})$ to $\bar{\otimes}^n E$ is continuous for all $n \geq 0$. By linearity there is no

loss of generality in assuming \underline{f} to be of the form

$(0, \dots, 0, f_r, 0, \dots)$, $f_r \in \bar{\otimes}^r E$, $r \geq 0$. Let $\{g_n^{(v)}\}$ be a net in $\bar{\otimes}^n E$

that converges to zero; by Proposition A.48 of Appendix A, $\{f_r \otimes g_n^{(v)}\}$

is a net that converges to zero in $\bar{\otimes}^{n+r} E$. Similarly we can prove

that the map $R(\underline{f}) : \underline{E} \longrightarrow \underline{E}$, $R(\underline{f})(\underline{g}) = \underline{g} \times \underline{f}$ is continuous.

Next we check the continuity of the involution. The involution is

continuous on $\bigoplus_{n=0}^{\infty} \bar{\otimes}_1^n E$ iff it is continuous on each $\bar{\otimes}_1^n E$ ($n \geq 0$). We

need only consider $n \geq 2$ and prove that $(f, g, \dots, \ell) \longmapsto \ell^* \otimes \dots \otimes g^* \otimes f^*$

$(f, g, \dots, \ell \in E)$ is separately continuous (see Proposition A.48 of

Appendix A). If $\{f^{(v)}\}$ is a net that tends to zero in E ,

$\{\ell^* \otimes \dots \otimes g^* \otimes f^{(v)*}\}$ is a net that tends to zero in $\bar{\otimes}_1^n E$. Evidently

this result is symmetric in the variables f, g, \dots, ℓ . Therefore the

involution is continuous on $\bigoplus_{n=0}^{\infty} \bar{\otimes}_1^n E$, and by continuity it can be

extended to its completion \underline{E} .

Finally, let us assume that $\sum_{i=1}^n \underline{f}^{i*} \times \underline{f}^i = 0$ and $\underline{f}^i \neq 0$ ($1 \leq i \leq n$). There

is no loss of generality in taking $\{\underline{f}^i\}_{1 \leq i \leq s}$, $1 \leq s \leq n$, as the subset of

$\{\underline{f}^i\}_{1 \leq i \leq n}$ whose elements have the greatest order for their last non-

vanishing components: call it $r \geq 0$. The case $r=0$ is trivial, so we

assume that $r \geq 1$. We then have that

$$\sum_{i=1}^s \underline{f}_r^{i*} \otimes \underline{f}_r^i = 0, \quad \underline{f}_r^i \in \bar{\otimes}^r E \quad (1 \leq i \leq s)$$

If the $\{\underline{f}_r^i\}_{1 \leq i \leq s}$ are linearly independent, then by the linear dis-

jointness of tensor products, this last equation implies that $\underline{f}_r^{i*} = 0$,

$1 \leq i \leq s$, which is a contradiction. Assume now that the $\{f_r^i\}_{1 \leq i \leq s}$ are linearly dependent and that $\{f_r^i\}_{1 \leq i \leq k}$, $1 \leq k \leq s$, is a maximal linearly independent subset of $\{f_r^i\}_{1 \leq i \leq s}$. There must then exist complex numbers λ_{ij} , $1 \leq i \leq k$, $k+1 \leq j \leq s$, such that $f_r^j = \sum_{i=1}^k \lambda_{ij} f_r^i$. Therefore

$$\sum_{i=1}^s f_r^{i*} \otimes f_r^i = \sum_{i=1}^k f_r^{i*} \otimes (f_r^i + \sum_{j=k+1}^s \sum_{\ell=1}^k \bar{\lambda}_{ij} \lambda_{\ell j} f_r^\ell) = 0$$

Now since the $\{f_r^{i*}\}_{1 \leq i \leq r}$ are linearly independent we use linear disjointness to get that

$$f_r^i + \sum_{j=k+1}^s \sum_{\ell=1}^k \bar{\lambda}_{ij} \lambda_{\ell j} f_r^\ell = 0 \quad (1 \leq i \leq k)$$

By the linear independence of the $\{f_r^i\}_{1 \leq i \leq k}$, we get that $1 + \sum_{j=k+1}^s |\lambda_{ij}|^2 = 0$, which is a contradiction. ■

We wish to point out that the inductive tensor product topology is the natural topology to use in BU-algebras. First of all, $E \bar{\otimes} F$ is but $E \hat{\otimes} F$ generally is not, barrelled when E and F are. Secondly, $E \bar{\otimes} F$ has transitive properties for inductive limits. Note that $\mathcal{D}(\mathbb{R}^n) \hat{\otimes} \mathcal{D}(\mathbb{R}^n)$ is the set $\mathcal{D}(\mathbb{R}^{2n})$ but with a strictly coarser topology than the canonical LF-topology.

Examples 1.9

- (a) $E = \mathcal{S}(\mathbb{R}^4)$ and $E = \mathcal{D}(\mathbb{R}^4)$ are the original BU-algebras due to Borchers [1] and Uhlmann [2] respectively. Here J is the identity.
- (b) For applications to the canonical anticommutation relations (CAR) and the canonical commutation relations (CCR) we take $E_{\mathbb{R}} = \mathcal{D}_{\mathbb{R}}(M) \oplus \mathcal{D}_{\mathbb{R}}(M)$ and $E = \mathcal{D}(M) \oplus \mathcal{D}(M)$, where the configuration space M is a paracompact C^∞ -manifold, countable at infinity and Hausdorff. The case $M = \mathbb{R}^n$ is typical. Here and hereafter \mathcal{D} will be equipped with its

canonical LF-topology. For ease of notation we shall write $E_{\mathbb{R}} = {}^2\mathcal{D}_{\mathbb{R}}(M)$ and $E = {}^2\mathcal{D}(M)$, so that $\underline{E} = \underline{{}^2\mathcal{D}}(M)$.

The involution J on $E_{\mathbb{R}}$ is taken to be $J(f,g) = (g,f)$

- (c) For any real finite dimensional Lie algebra \mathcal{G} , the BU-algebra formed from $E_{\mathbb{R}} = {}^2\mathcal{D}_{\mathbb{R}}(M, \mathcal{G}) \simeq {}^2\mathcal{D}_{\mathbb{R}}(M) \otimes \mathcal{G}$ with involution $J(f(x) \otimes X) = -f(x) \otimes X$, will be shown to be pertinent to the description of the current algebra based on \mathcal{G} .

1.3 The \mathcal{N} Topology

The isomorphisms given by the following Proposition will be of some use later.

Proposition 1.10 Let \mathcal{A} be an I^* -algebra, \mathcal{T} its initial topology and \mathcal{N} the finest locally convex topology on \mathcal{A} for which the linear map $f \otimes g \mapsto fg$ from $\mathcal{A}[\mathcal{T}] \otimes_{\pi} \mathcal{A}[\mathcal{T}]$ onto \mathcal{A} is continuous.

Denote the kernel of this map by K . Then

$$\mathcal{A}[\mathcal{T}] \otimes_{\pi} \mathcal{A}[\mathcal{T}] / K \simeq \mathcal{A}[\mathcal{N}] \quad (1.3)$$

$$\mathcal{A}[\mathcal{T}] \otimes_1 \mathcal{A}[\mathcal{T}] / K \simeq \mathcal{A}[\mathcal{T}] \quad (1.4)$$

and \mathcal{T} is finer than \mathcal{N} .

Proof We will only prove the second isomorphism, since the other case

is similar. Let \mathcal{U} be the finest locally convex topology on \mathcal{A} such that the map $M : \mathcal{A}[\mathcal{T}] \otimes_1 \mathcal{A}[\mathcal{T}] \rightarrow \mathcal{A}$, $M(f \otimes g) = fg$ is continuous.

Then the map $m : \mathcal{A}[\mathcal{T}] \times \mathcal{A}[\mathcal{T}] \rightarrow \mathcal{A}[\mathcal{U}]$,

$m((f,g)) = fg$, is separately continuous and so the maps $L(f)(g) = fg$

from $\mathcal{A}[\mathcal{T}]$ into $\mathcal{A}[\mathcal{U}]$ are continuous. Taking $f=1$ implies that

\mathcal{T} is finer than \mathcal{U} , but by definition \mathcal{U} is finer than \mathcal{T} , so

$\mathcal{U} = \mathcal{T}$.

Going to the quotient, it is obvious that

$$\tilde{M} : A[\mathcal{T}] \otimes_1 A[\mathcal{T}] / K \rightarrow A[\mathcal{T}]$$

is continuous and one-to-one.

Now $\tilde{M}^{-1} \circ M$ is the canonical projection from $A[\mathcal{T}] \otimes_1 A[\mathcal{T}]$ onto $A[\mathcal{T}] \otimes_1 A[\mathcal{T}] / K$, which is continuous. But as $\mathcal{T} = \mathcal{U}$, $\tilde{M}^{-1} \circ M$ is continuous iff \tilde{M}^{-1} is. Thus \tilde{M} furnishes the indicated isomorphism.

The last assertion follows from the fact that the ι -topology is finer than the π -topology : Appendix A, Proposition A.43. ■

The topology \mathcal{W} was introduced for $\underline{\mathcal{F}}(\mathbb{R}^n)$ by Yngvason in [17].

For BU-algebras one can give an easily verifiable necessary and sufficient condition for the equality of the topologies \mathcal{T} and \mathcal{W} . But before doing this we need to prove a technical Lemma, quoted without proof in [18].

Lemma 1.11 Let E be a non-normable metrizable locally convex space. Then there is a generating family $\{p_k\}_{k \geq 1}$ consisting of increasing mutually inequivalent seminorms and a family $\{\phi_k\}_{k \geq 1}$ of continuous linear functions such that $|\phi_k(f)| \leq c_k p_{k+1}(f)$, for all $f \in E$, $k \geq 1$, $0 < c_k < +\infty$ and $\sup_{f \in B_{p_k}} |\phi_k(f)| = +\infty$, $k \geq 1$, where B_{p_k} is the open unit ball associated to the seminorm p_k .

Proof Since E is metrizable and non-normable there is a generating family $\{p'_k\}_{k \geq 1}$ consisting of increasing mutually inequivalent seminorms. Define $p_1 = p'_1$. Since B_{p_1} is unbounded, there is a continuous linear functional ϕ_1 , such that $\sup_{f \in B_{p_1}} |\phi_1(f)| = +\infty$. The continuity of ϕ_1 implies the existence of $n_1 \geq 2$, and c_1 ($0 < c_1 < +\infty$)

such that $|\phi_1(f)| \leq c_1 p_{n_1}'(f)$, for all $f \in E$. We define $p_2 = p_{n_1}'$ and the construction then proceeds in an obvious way by induction. ■

Proposition 1.12 For a BU-algebra \underline{E} , $\mathcal{N} = \mathcal{T}$ iff E is an LB-space.

Proof Let E be an LB-space. As E is also nuclear, the spaces $\overline{\otimes}^n E$, $n \geq 2$, are LB (c.f. Appendix A, Proposition A.49). This implies that \underline{E} is an LB-space (c.f. ibid, Proposition A.50) and by Proposition A.43 we have $\underline{E} \otimes_1 \underline{E} = \underline{E} \otimes_\pi \underline{E}$. Therefore by Proposition 1.10, $\mathcal{N} = \mathcal{T}$.

Now let E be an LF-space, that is not LB, and $\{E_k\}_{k \geq 1}$ a sequence of definition of E . There exists an $\ell \geq 1$ such that E_ℓ is non-normable.

Let $\{\phi_k\}_{k \geq 1}$ be a family of continuous linear functionals on E_ℓ obeying the conditions of Lemma 1.11. We show that the bilinear form on

\underline{E}_ℓ , $(\underline{f}, \underline{g}) \mapsto \phi(\underline{f} \times \underline{g})$, is not jointly continuous and hence, since ϕ is continuous, that $(\underline{f}, \underline{g}) \mapsto \underline{f} \times \underline{g}$ is not jointly continuous. Assume there is a continuous seminorm p on \underline{E}_ℓ such that $|\phi(\underline{f} \times \underline{g})| \leq p(\underline{f})p(\underline{g})$.

Taking $\underline{f} = (0, f_1, 0, \dots)$, $\underline{g} = \underline{1}$, and $\underline{g} = (0, \dots, g_n, 0, \dots)$, ($n \geq 1$), with $(\phi_{n+1} \otimes \dots \otimes \phi_{n+1})(g_n) \neq 0$ we get $|\phi_k(f_1)| \leq c_k p(f_1)$, where $c_1 = p(1)$ and $c_k = p(g_{k-1}) |(\phi_k \otimes \dots \otimes \phi_k)(g_{k-1})|^{-1}$ for $k \geq 2$, which is a contradiction since the ϕ_k cannot all be dominated by a single continuous seminorm.

Since E is nuclear, $\overline{\otimes}^n_E$ induces the original topology on $\overline{\otimes}^n_{E_\ell}$ for all $n \geq 1$ (c.f. Proposition A.49) and consequently \underline{E} induces the original topology on \underline{E}_ℓ . Clearly, then, the product in \underline{E} cannot be jointly continuous either, i.e., $\mathcal{N} \neq \mathcal{T}$. ■

Corollary 1.13 For a BU-algebra \underline{E} , the product is jointly continuous iff E is an LB-space.

We remark that, since a Banach space is nuclear iff it is finite dimensional, every sequence of definition of a nuclear LB-space consists of finite dimensional spaces.

In [17] Yngvason proved that the multiplication in $\underline{\mathcal{L}}(\mathbb{R}^n)$ is not jointly continuous.

CHAPTER 2

ORDER PROPERTIES

The order properties of $\mathcal{F}(\mathbb{R}^n)$ are well known [3,4,19]. In this section we consider the corresponding properties for general I^* -algebras.

2.1 Ordered I^* -Algebras

Definition 2.1

(a) For an I^* -algebra \mathcal{A} the set of hermitian elements is

$$\mathcal{A}_h = \{f \in \mathcal{A} : f^* = f\}, \quad (2.1a)$$

the set of positive elements is

$$\mathcal{A}_+ = \{\sum_{i=1}^n f_i^* f_i : f_i \in \mathcal{A}, n \in \mathbb{N}\}, \quad (2.1b)$$

and the closure of \mathcal{A}_+ is written $\overline{\mathcal{A}_+}$.

(b) The subsets of the dual \mathcal{A}' ,

$$\mathcal{A}'_h = \{\phi \in \mathcal{A}' : \phi(f^*) = \overline{\phi(f)} \text{ for all } f \in \mathcal{A}\}, \quad (2.2a)$$

$$\mathcal{A}'_+ = \{\phi \in \mathcal{A}' : \phi(f) \geq 0 \text{ for all } f \in \mathcal{A}_+\}, \quad (2.2c)$$

$$\text{and } \mathbb{E}(\mathcal{A}) = \{\phi \in \mathcal{A}'_+ : \phi(1) = 1\} \quad (2.2c)$$

will be known as the set of hermitian functionals, positive functionals and states, respectively.

Lemma 2.2 A positive functional is hermitian and obeys the Cauchy-Schwarz inequality

$$|\phi(f^* g)|^2 \leq \phi(f^* f) \phi(g^* g) \quad (f, g \in \mathcal{A}) \quad (2.3)$$

Proof See [11], Lemma I.4.2. ■

Proposition 2.3 The hermitian part, \mathcal{A}_h , of an I^* -algebra \mathcal{A} is a complete real vector space whose complexification is \mathcal{A} . The cones

\mathcal{A}_+ , $\overline{\mathcal{A}}_+$ are proper strict b-cones which are generating for \mathcal{A}_h .

The hermitian functionals \mathcal{A}'_h constitute a complete real vector space whose complexification is \mathcal{A}' , with $\mathcal{A}'_h = (\mathcal{A}_h)'$. The cone \mathcal{A}'_+ is a complete proper normal cone with base $E(\mathcal{A})$. The set $\mathcal{A}'_+ - \mathcal{A}'_+$ is dense in \mathcal{A}'_h .

Proof The completeness of \mathcal{A}_h follows from the continuity of the involution and the completeness of \mathcal{A} . The remaining properties of \mathcal{A}_h are obvious. For the assertions about \mathcal{A}_+ we follow [19]. Since $f = \frac{1}{4}(1+f)^2 - \frac{1}{4}(1-f)^2$, for all $f \in \mathcal{A}_h$, \mathcal{A}_+ is generating for \mathcal{A}_h . Let $B \subset \mathcal{A}_h$ be bounded; by Proposition 1.3 the sets $B' = \{\frac{1}{4}(1+f)^2 : f \in B\}$ and $B'' = \{\frac{1}{4}(1-f)^2 : f \in B\}$ are bounded. The set $B_1 = B' \cup B''$ is also bounded and $B \subset B_1 \cap \mathcal{A}_+ - B_1 \cap \mathcal{A}_+$. Therefore \mathcal{A}_+ is a strict b-cone. \mathcal{A}_+ is proper because $\sum_{i=1}^n f_i^* f_i = 0$ implies $f_i = 0$ ($1 \leq i \leq n$). We now come to the properties of $\overline{\mathcal{A}}_+$: $\overline{\mathcal{A}}_+ \supset \mathcal{A}_+$ and \mathcal{A}_+ generating implies that $\overline{\mathcal{A}}_+$ is generating; $\overline{\mathcal{A}}_+$ is a proper strict b-cone because the strong topology on \mathcal{A}'_h is compatible with the duality $\langle \mathcal{A}_h, \mathcal{A}'_h \rangle$ and \mathcal{A}_+ is a proper strict b-cone ([20], Cor 1.23, p74).

The properties of \mathcal{A}'_h follow from the fact that $\phi^*(f) = \overline{\phi(f^*)}$ defines a continuous involution $\phi \mapsto \phi^*$ on \mathcal{A}' . \mathcal{A}'_+ is a proper normal cone because $\overline{\mathcal{A}}_+$ is a proper generating strict b-cone and \mathcal{A}_h is reflexive ([20], Cor 1.26, p75). \mathcal{A}'_+ is complete because it is closed and \mathcal{A}'_h is complete. The assertion about the base follows from the Cauchy-Schwartz inequality. Finally since $\overline{\mathcal{A}}_+$ is a proper cone, $\mathcal{A}'_+ - \mathcal{A}'_+$ is dense in \mathcal{A}'_h ([20], Property 1.19, p71). ■

We are now in a position to prove further properties of the topology \mathcal{W} .

Proposition 2.4 For any I^* -algebra \mathcal{A} , $\mathcal{A}[\mathcal{N}]$ is Hausdorff and nuclear. The involution is continuous from $\mathcal{A}[\mathcal{N}]$ to $\mathcal{A}[\mathcal{N}]$.

Proof First we follow [17] to prove that $\mathcal{A}'_+ \subset \mathcal{A}[\mathcal{N}]'$. Since the involution is continuous on $\mathcal{A}[\mathcal{T}]$, \mathcal{T} is generated by a family of $*$ -symmetric seminorms $\{p_\alpha\}_{\alpha \in \Lambda}$, i.e. $p_\alpha(f) = p_\alpha(f^*)$, for all $f \in \mathcal{A}$, $\alpha \in \Lambda$. From equation (1.3) and Proposition A.51 of Appendix A we get that the seminorm

$$\hat{p}_\alpha(f) = \inf\{\sum_{i=1}^n p_\alpha(g_i)p_\alpha(h_i) : \sum_{i=1}^n g_i h_i = f, n \in \mathbb{N}\} \quad (2.4)$$

is \mathcal{N} -continuous. Now since $\mathcal{A}[\mathcal{T}]$ is barrelled $f \mapsto \phi(f^*f)^{1/2}$ is a continuous seminorm for all $\phi \in \mathcal{A}'_+$: [21], Theorem 4.1. Then there is an $\alpha \in \Lambda$ such that $\phi(f^*f)^{1/2} \leq p_\alpha(f)$ for all $f \in \mathcal{A}$, and from the Cauchy-Schwarz inequality we get

$$|\phi(\sum_{i=1}^n g_i h_i)| \leq \sum_{i=1}^n p_\alpha(g_i)p_\alpha(h_i).$$

Therefore $|\phi(f)| \leq \hat{p}_\alpha(f)$, i.e., $\phi \in \mathcal{A}[\mathcal{N}]'$.

We now prove the topological properties of \mathcal{N} . Since $\mathcal{A}'_+ - \mathcal{A}'_+$ is dense in \mathcal{A}'_h (Proposition 2.3), \mathcal{A}'_+ separates the points of \mathcal{A} . Then $\mathcal{A}'_+ \subset \mathcal{A}[\mathcal{N}]'$ implies that $\mathcal{A}[\mathcal{N}]$ is Hausdorff. It follows from this that K is closed in $\mathcal{A}[\mathcal{T}] \otimes_\pi \mathcal{A}[\mathcal{T}]$ (Proposition 1.10) and from equation (1.3) that $\mathcal{A}[\mathcal{N}]$ is nuclear.

Finally the involution is continuous on $\mathcal{A}[\mathcal{N}]$ because $\{\hat{p}_\alpha\}_{\alpha \in \Lambda}$ is a generating family of seminorms for \mathcal{N} and the \hat{p}_α are $*$ -symmetric. ■

The next proposition gives an interesting relationship between topological and order properties of I^* -algebras.

Proposition 2.5 Every bounded set in $A_h[\mathcal{T}]$ is order-bounded.

Proof Since $A_h[\mathcal{T}]'$ is reflexive, its topology is defined by the family of seminorms $p_B(\phi) = \sup_{f \in B} |\phi(f)|$, as B varies over all bounded subsets of $A_h[\mathcal{T}]$. By [22], Theorem 3, (see also Appendix B, Proposition B.18) since A_+' is a normal cone in a nuclear and barrelled space $A_h[\mathcal{T}]'$, there is a summable sequence of positive numbers $\{\lambda_n^B\}_{n \in \mathbb{N}}$ and a bounded sequence $\{f_n^B\}$ in \bar{A}_+ such that

$$p_B(\phi) \leq \sum_{n \in \mathbb{N}} \lambda_n^B |\phi(f_n^B)| \quad \text{for all } \phi \in A_h[\mathcal{T}]' \quad (2.5)$$

If $\phi \in A_+'$ and $B \subset \bar{A}_+$, we then get

$$\phi(\sum_{n \in \mathbb{N}} \lambda_n^B f_n^B - f) \geq 0 \quad \text{for all } f \in B$$

Thus B is contained in the order interval $[0, \sum_{n \in \mathbb{N}} \lambda_n^B f_n^B]$, i.e., it is order bounded. Now if B is an arbitrary bounded subset of $A_h[\mathcal{T}]$, there is a bounded set $B_1 \subset \bar{A}_+$ such that $B \subset B_1 - B_1$, since \bar{A}_+ is a strict b-cone. Then $B \subset [-g_{B_1}, g_{B_1}]$ where $B_1 \subset [0, g_{B_1}]$, $g_{B_1} \in \bar{A}_+$. ■

In [17] Yngvason proved Proposition 2.5 for $\mathcal{L}(\mathbb{R}^n)$ by a different method.

2.2 Property (N)

We have found that the following condition on the \mathcal{N} -topology enables us to prove a number of further order properties. We leave open the question of whether or not the condition follows from the axioms.

An I^* -algebra A has property (N) if the convergence of the net $\{\sum_{i=1}^{n_v} f_i^{(v)*} f_i^{(v)}\}_{v \in \Gamma}$ to zero in $A_h[\mathcal{N}]$ implies the convergence of the net $\{\sum_{i=1}^{n_v} f_i^{(v)*} \otimes f_i^{(v)}\}_{v \in \Gamma}$ to zero in $A[\mathcal{T}] \otimes_{\pi} A[\mathcal{T}]$.

It can be shown that property (N) is equivalent to its apparently

weaker version obtained by replacing \mathcal{W} by $\sigma(A_h, A'_+ - A'_+)$.

Proposition 2.6 The topology \mathcal{T} of an I^* -algebra A is given by its states, i.e., has $\{f \mapsto \phi(f^*f)^{1/2} : \phi \in \mathbb{E}(A)\}$ as a generating family of seminorms, iff A has property (N).

Proof Let p be a continuous seminorm on $A[\mathcal{T}]$. By nuclearity there is a summable sequence of positive numbers $\{\lambda_m\}_{m \in \mathbb{N}}$ and a \mathcal{T} -equi-continuous sequence of linear functionals $\{T_m\}_{m \in \mathbb{N}}$ such that

$$p(f)^2 \leq \sum_{m \in \mathbb{N}} \lambda_m |T_m(f)|^2 \quad (2.6)$$

for all $f \in A$ ([14]; see also Appendix A).

If $\{p_\alpha\}_{\alpha \in \Lambda}$ is the generating family of seminorms introduced in the proof of Proposition 2.4, then there exists a $\beta \in \Lambda$ such that

$$\sum_{i=1}^n p(f_i)^2 \leq ||\lambda||_1 (p_\beta \otimes_\epsilon p_\beta) (\sum_{i=1}^n f_i^* \otimes f_i) \quad (2.7)$$

(see Appendix A, Proposition A.51, for the definition of $p_\beta \otimes_\epsilon p_\beta$). To see this, first pick $\beta \in \Lambda$ such that $\{T_m\}_{m \in \mathbb{N}} \subset B_{p_\beta}^\circ$. By $*$ -symmetry and the Cauchy-Schwarz inequality,

$$(p_\beta \otimes_\epsilon p_\beta) (\sum_{i=1}^n f_i^* \otimes f_i) = \sup_{\phi \in B_{p_\beta}^\circ} \sum_{i=1}^n |\phi(f_i)|^2.$$

As $T_m \in B_{p_\beta}^\circ$, $\sum_{i=1}^n |T_m(f_i)|^2 \leq (p_\beta \otimes_\epsilon p_\beta) (\sum_{i=1}^n f_i^* \otimes f_i)$, giving, the desired inequality. By Proposition A.51 we have $p_\beta \otimes_\epsilon p_\beta \leq p_\beta \otimes_\pi p_\beta$ and therefore

$$\sum_{i=1}^n p(f_i)^2 \leq ||\lambda||_1 (p_\beta \otimes_\pi p_\beta) (\sum_{i=1}^n f_i^* \otimes f_i), \quad (2.8)$$

By a theorem of Ky Fan (Appendix B, Proposition B.20) there is an $\omega \in A_h[\mathcal{W}]'$ such that

$$\omega(f^*f) \geq p(f)^2 \quad \text{for all } f \in A$$

iff $\lim_{v \rightarrow 0} \sum_{i=1}^n f_i^{(v)*} f_i^{(v)} \rightarrow 0$ in $A_h[\mathcal{W}]$ and $\lim_{v \rightarrow 0} \sum_{i=1}^n p(f_i^{(v)})^2 \rightarrow \xi$,

implies $\xi = 0$. If property (N) holds, this last condition is guaranteed by inequality (2.8). Since the involution is continuous on $\mathcal{A}[\mathcal{N}]$ (Proposition 2.4),

$$\phi(f) = \frac{1}{2} \omega(f+f^*) + \frac{i}{2} \omega(i(f^*-f)) \quad (2.9)$$

is in $\mathcal{A}[\mathcal{N}]'$ and $\phi(f^*f) \geq p(f)^2$. It remains to prove the continuity of the seminorm $f \mapsto \phi(f^*f)^{1/2}$, but this follows as in Proposition 2.4.

Assume now that the topology \mathcal{T} is given by the states. If ϕ_α is the state that dominates p_α we get (c.f. Proposition A.51)

$$(p_\alpha \otimes_{\pi} p_\alpha) \left(\sum_{i=1}^n f_i^* \otimes f_i \right) \leq \sum_{i=1}^n p_\alpha(f_i)^2 \leq \phi_\alpha \left(\sum_{i=1}^n f_i^* f_i \right)$$

Property (N) then follows from the inequality involving $p_\alpha \otimes_{\pi} p_\alpha$ and ϕ_α , and the fact that $\mathcal{A}'_+ \subset \mathcal{A}[\mathcal{N}]'$ (Proposition 2.4). ■

Corollary 2.7 Let \mathcal{A} have property (N). Then $\overline{\mathcal{A}}_+$ has a base iff there is a continuous norm on $\mathcal{A}[\mathcal{T}]$.

Proof Now $\overline{\mathcal{A}}_+$ has a base iff there is a strictly positive linear functional on \mathcal{A} : [20], Proposition 3.6 (p26). Let $f \mapsto ||f||$ be the hypothesized norm. By Proposition 2.6 there is a state ϕ that dominates $||\cdot||$, so $0 < ||f|| \leq \phi(f^*f)^{1/2}$ if $f \neq 0$. ■

In [17] Yngvason proved that the topology of $\mathcal{L}(\mathbb{R}^n)$ is given by its states.

The next Proposition shows that the \mathcal{N} topology is better adapted than \mathcal{T} for the study of order properties.

Proposition 2.8 Let \mathcal{A} have property (N). Then $\overline{\mathcal{A}}_+$ is normal in $\mathcal{A}_h[\mathcal{N}]$. If, further, \mathcal{A} is a BU-algebra with $\mathcal{T} \neq \mathcal{N}$, then $\overline{\mathcal{A}}_+$ is not normal in $\mathcal{A}_h[\mathcal{T}]$.

Proof The first assertion follows as in [17], Theorem 4. For the second, one can show as in [17], Theorem 5, that $\phi \in \mathcal{A}'_+ - \mathcal{A}'_+$ iff $(f, g) \rightsquigarrow \phi(fg)$ is a jointly continuous bilinear form. In the proof of Proposition 1.12 we have shown that when $\mathcal{T} \neq \mathcal{N}$ there is a $\phi \in \mathcal{A}[\mathcal{T}]'$ such that $(f, g) \rightsquigarrow \phi(fg)$ is not jointly continuous. Therefore at least one of ^{the} hermitian functionals $\phi + \phi^*$, $i(\phi - \phi^*)$, is not in $\mathcal{A}'_+ - \mathcal{A}'_+$, i.e., $\mathcal{A}'_+ - \mathcal{A}'_+ \neq \mathcal{A}_h[\mathcal{T}]'$. The conclusion follows from Proposition B.16 of Appendix B. ■

The following proposition shows that in most cases of interest the topological properties of \mathcal{N} are not as rich as those of \mathcal{T} .

Proposition 2.9 Let \mathcal{A} have property (N). If $\mathcal{T} \neq \mathcal{N}$, then \mathcal{N} is not barrelled. If, further, \mathcal{N} is complete, then it is not bornological, it has the same bounded sets as \mathcal{T} , and $\overline{\mathcal{A}}_+$ is generated by its extreme rays.

Proof If ϕ is a state, $f \rightsquigarrow \phi(g^*fg)$ is again a state, and therefore \mathcal{N} -continuous (c.f. Proposition 2.4). A polarization argument then shows that $f \rightsquigarrow \phi(gf)$ is also \mathcal{N} -continuous for any state ϕ , and all $g \in \mathcal{A}$.

From the Cauchy-Schwarz inequality it follows that for any state ϕ

$$\phi(f^*f) = \sup_g \{ |\phi(g^*f)| : \phi(g^*g) = 1 \}. \quad (2.10)$$

we may write, therefore,

$$\{f \in \mathcal{A} : \phi(f^*f)^{1/2} \leq 1\} = \bigcap_g \{f \in \mathcal{A} : |\phi(g^*f)| \leq 1, \phi(g^*g) = 1\}. \quad (2.11)$$

From the \mathcal{N} -continuity of $f \mapsto \phi(gf)$ it follows that the closed unit ball of the seminorm $f \mapsto \phi(f^*f)^{1/2}$ is \mathcal{N} -closed, and hence an \mathcal{N} -barrel. As $\mathcal{T} \neq \mathcal{N}$, there is a state ϕ whose closed unit ball is a \mathcal{T} -neighbourhood of zero (Proposition 2.6) but not an \mathcal{N} -neighbourhood of zero; hence \mathcal{N} is not barrelled.

As a complete bornological space is barrelled : [12], Corollary, (p63), if \mathcal{N} is complete it is not bornological.

Since \mathcal{T} has a basis of neighbourhoods of zero consisting of \mathcal{N} -barrels and \mathcal{N} is complete, every \mathcal{N} -bounded set is \mathcal{T} -bounded : [13], Lemma 36.2, [15], p109. The reverse inclusion is obvious.

Finally since $\mathcal{A}[\mathcal{T}]$ is complete and dual nuclear, a theorem of Thomas (Appendix B, Proposition B.19) will imply the conclusion about the extreme rays if we show that the order intervals $[0, f]$ ($f \in \overline{\mathcal{A}}_+$) are \mathcal{T} -compact. But as $\mathcal{A}[\mathcal{T}]$ is Montel and $[0, f]$, \mathcal{T} -closed, we only need check \mathcal{T} -boundedness of these order intervals. As $\overline{\mathcal{A}}_+$ is normal in $\mathcal{A}_h[\mathcal{N}]$ (Proposition 2.8), $[0, f]$ is \mathcal{N} -bounded : [12], Corollary 2 (p216) and therefore \mathcal{T} -bounded. ■

Part of the information contained in the next Proposition is relevant for the problem of extension of positive linear functionals.

Proposition 2.10

- (a) If ϕ is an element of the algebraic dual of \mathcal{A} that takes positive values on $\overline{\mathcal{A}}_+$, then ϕ is continuous.
- (b) If \mathcal{A} is a BU-algebra, $\overline{\mathcal{A}}_+$ has empty interior.
- (c) If $\mathcal{A} \neq \mathbb{C}$ has property (N), $\overline{\mathcal{A}}_+$ has empty interior.

Proof For (a) and (b) see [19], Theorem 3.1, and Lemma 2.4 respectively.

For (c) we note that if \bar{A}_+ were to have interior points, there would be order units for A_h : [12] exercise 10(a) (p251). But \mathcal{W} is a non-normable Hausdorff topology (Corollary 1.6 and Proposition 2.4) in which \bar{A}_+ is normal (Proposition 2.8), therefore A_h has no order units : [12] exercise 10(c) (p252). ■

CHAPTER 3

REPRESENTATIONS OF THE COMMUTATION RELATIONS AND OF CURRENT ALGEBRA

3.1 The GNS Construction

Every state on an I^* -algebra determines a strongly cyclic representation of the algebra as a $*$ -operator family. For BU-algebras, this corresponds to the Wightman reconstruction [23]; in all cases it is the GNS construction [16].

Theorem 3.1

(a) Let ϕ be a state on the I^* -algebra \mathcal{A} , and

$$L(\phi) = \{f \in \mathcal{A} : \phi(f^*f) = 0\} \quad (3.1)$$

its left kernel. Then $L(\phi) \subset \mathcal{A}$ is a closed left ideal.

Equipped with the sesquilinear form

$$\langle [f]_\phi, [g]_\phi \rangle = \phi(f^*g) \quad (3.2)$$

the linear space $\mathcal{D}_\phi \equiv \mathcal{A}/L(\phi)$ is a pre-Hilbert space, where

$f \mapsto [f]_\phi$ indicates the canonical projection from \mathcal{A} to \mathcal{D}_ϕ .

The linear extension of the mapping $f \mapsto \pi_\phi(f)$

$$\pi_\phi(f)[g]_\phi = [fg]_\phi \quad (3.3)$$

affords a strongly cyclic $*$ -representation of \mathcal{A} as (usually unbounded) operators on the common dense domain \mathcal{D}_ϕ . The cyclic vector $\Omega_\phi = [1]_\phi$ is the "ground state" or "vacuum" for ϕ . The $*$ -symmetry is given by $\pi_\phi(f)^* \supset \pi_\phi(f^*)$, so the $\pi_\phi(f)$ are closeable.

(b) Consider the BU-algebra \underline{E} and let there be given a Hilbert space \mathcal{H} , a dense linear subspace $\mathcal{D} \subset \mathcal{H}$ and a linear mapping, π , from E into (usually unbounded) linear operators on \mathcal{H} such that (i) $\text{dom}[\pi(f)] \supset \mathcal{D}$, (ii) $\pi(f)\mathcal{D} \subset \mathcal{D}$,

(iii) $\langle \Psi, \pi(f)\Phi \rangle = \langle \pi(f^*)\Psi, \Phi \rangle$ on \mathcal{D} ,

(iv) $f \mapsto \langle \Psi, \pi(f)\Phi \rangle$ is in E' , for all $\Phi, \Psi \in \mathcal{D}$.

Then every normalized vector $\Omega \in \mathcal{D}$ defines a state

$\phi_\Omega = \{\phi_n : n+1 \in \mathbb{N}\}$ on \underline{E} by continuous linear extension from

$$\phi_n(f_1 \otimes f_2 \otimes \dots \otimes f_n) = \langle \Omega, \pi(f_1)\pi(f_2)\dots\pi(f_n)\Omega \rangle \quad (3.4)$$

(c) With the hypotheses and notation of (b), let \mathcal{D}_Ω be the linear span

$$V\{\Omega, \pi(f)\Omega, \pi(g)\pi(h)\Omega, \dots, f, g, h, \dots \in E\}$$

and π_Ω the restriction of π to \mathcal{D}_Ω . Then $(\pi_\Omega, \mathcal{D}_\Omega)$ is a $*$ -operator family representing \underline{E} which is unitarily equivalent to the GNS construction from ϕ_Ω .

Proof See [11], Theorem I.4.5. ■

3.2 Representations of the CAR and the CCR

Definition 3.2 To any $*$ -representation (π, \mathcal{D}) of the field algebra ${}^2\mathcal{D}(M)$ we associate the complex linear fields

$$\begin{aligned} a^*(f) &= \pi(f, 0) \\ a(f) &= \pi(0, f) \end{aligned} \quad (f \in \mathcal{D}(M)) \quad (3.5)$$

We note that as π is a $*$ -representation, it follows immediately that the fields (a, a^*) satisfy the $*$ -symmetry relations

$$a(f)^* = a^*(\bar{f}), \quad a^*(f)^* = a(\bar{f}) \quad (3.6)$$

where \bar{f} is the complex conjugate of $f \in \mathcal{D}(M)$.

From Definition 3.2 and part (b) of Theorem 3.1 we see that, for a cyclic representation, knowledge of the pair (a, a^*) determines a state on ${}^2\mathcal{D}(M)$. More precisely we have the following.

Proposition 3.3 Given a Hilbert space \mathcal{H} , a dense linear subspace $\mathcal{D} \subset \mathcal{H}$ and a pair (a, a^*) of complex linear mappings from $\mathcal{D}(M)$ into (unbounded) linear operators on \mathcal{H} such that

- (i) $\text{dom}[a^\#(f)] \supset \mathcal{D}$
- (ii) $a^\#(f)\mathcal{D} \subset \mathcal{D}$
- (iii) equation (3.6)
- (iv) $f \mapsto \langle \Psi, a^\#(f)\Phi \rangle$ is in $\mathcal{D}(M)'$ for all $\Psi, \Phi \in \mathcal{D}$, where $a^\#(f) = a(f)$ or $a^*(f)$.

Then every normalized vector $\Omega \in \mathcal{D}$ defines a state $\phi_\Omega = \{\phi_n : n \in \mathbb{N}\}$ on $\mathcal{D}(M)$ by continuous linear extension from

$$\phi_n(F_1 \otimes \dots \otimes F_n) = \langle \Omega, [a^*(f_1) + a(g_1)] \dots [a^*(f_n) + a(g_n)] \Omega \rangle \quad (3.7)$$

where $F_k = (f_k, g_k)$, and $f_k, g_k \in \mathcal{D}(M)$ ($1 \leq k \leq n$). ■

The CAR or CCR are defined with respect to some symmetric bilinear form

$\langle \cdot | \cdot \rangle$ on $\mathcal{D}_\mathbb{R}(M)$. Usually this will be the inner product on $L^2(M)$ restricted to $\mathcal{D}_\mathbb{R}(M)$. In general we shall demand that $\langle \cdot | \cdot \rangle$ be non-degenerate and jointly continuous. Moreover we shall suppose that whenever $f, g \in \mathcal{D}_\mathbb{R}(M)$ have disjoint supports, $\langle f | g \rangle = 0$.

Proposition 3.4 Let I_ϵ ($\epsilon \pm 1$) be the smallest closed $*$ -ideal generated by elements of the form

$$\begin{aligned} & (0, 0, (f, 0) \otimes (g, 0) + \epsilon(g, 0) \otimes (f, 0), 0, \dots) \\ & (0, 0, (0, f) \otimes (0, g) + \epsilon(0, g) \otimes (0, f), 0, \dots) \quad (f, g \in \mathcal{D}_\mathbb{R}(M)) \\ & (-\langle f | g \rangle, 0, \epsilon(g, 0) \otimes (0, f) + (0, f) \otimes (g, 0), 0, \dots) \end{aligned} \quad (3.8)$$

Any state ϕ on $\mathcal{D}(M)$ which annihilates I_ϵ gives rise to fields (a, a^*) satisfying

$$\begin{aligned} [a^\#(f), a^\#(g)]_\epsilon \phi &= 0 \\ [a(f), a^*(g)]_\epsilon \phi &= \langle f | g \rangle \phi \end{aligned} \quad (3.9)$$

for all $\phi \in \mathcal{D}_\phi$, $f, g, \in \mathcal{D}_{\mathbb{R}}(M)$

Conversely if fields are given satisfying the hypothesis of Proposition 3.3 and the relations (3.9), then the state ϕ_Ω annihilates I_ϵ .

Proof As in [11], Theorem I.5.6. ■

For obvious reasons, a state annihilating I_+ will be termed a CAR state, and one which annihilates I_- will be termed a CCR state.

By a well known theorem, e.g. [24], Theorem (p270), the fields $a^\#(f)$ associated with a CAR state are bounded : it follows immediately that

$$a(f)^* = a^*(\bar{f}) \quad a^*(f)^* = a(\bar{f}) \quad (3.10)$$

Thus a CAR state on ${}^2\mathcal{D}(M)$ leads to a representation of the CAR in the usual sense [24,25].

On the other hand the CCR fields are not bounded, so the symmetry relations (3.10) are generally the most that can be said. The relation between CCR states on ${}^2\mathcal{D}(M)$ and Weyl algebra states is therefore a delicate one. We have not found any natural condition on the CCR states of ${}^2\mathcal{D}(M)$ so that the fields $Q(f) = 2^{-\frac{1}{2}}[a(f) + a^*(f)]$, $P(f) = i2^{-\frac{1}{2}}[a^*(f) - a(f)]$, $f \in \mathcal{D}_{\mathbb{R}}(M)$, are essentially self-adjoint and their exponentials satisfy the Weyl relations. Going the other way one can show, using results of Hegerfeldt [26] on Garding domains, that every state on the Weyl algebra determines a CCR state on ${}^2\mathcal{D}(M)$.

For completeness we mention that the graded structure of BU-algebras allows the possibility of defining a symmetric product of states [11].

Definition 3.5

If ϕ and ψ are states on the BU-algebra \underline{E} we define a new state $\omega \equiv \phi \otimes \psi$ on \underline{E} by the formula

$$\omega_n(f_{i_1} \otimes \dots \otimes f_{i_r} \otimes f_{i_{r+1}} \otimes \dots \otimes f_{i_n}) = \sum_{\text{partitions}} \phi_r(f_{i_1} \otimes f_{i_2} \otimes \dots \otimes f_{i_r}) \times \psi_{n-r}(f_{i_{r+1}} \otimes f_{i_{r+2}} \otimes \dots \otimes f_{i_n}) \quad (3.11)$$

where the sum is over all partitions $\{i_1, i_2, \dots, i_r\}, \{i_{r+1}, i_{r+2}, \dots, i_n\}$ of $\{1, 2, \dots, n\}$ such that

$$i_1 < i_2 < \dots < i_r, i_{r+1} < i_{r+2} < \dots < i_n, \text{ and } \{i_1, i_2, \dots, i_r\} \cap \{i_{r+1}, i_{r+2}, \dots, i_n\} = \emptyset$$

Proposition 3.6 The state $\omega = \phi \otimes \psi$ coincides with the vector state

$\Omega_\phi \otimes \Omega_\psi$ of the field

$$\{\pi_\phi(f) \otimes 1_{\psi} + 1_{\phi} \otimes \pi_\psi(f), \mathcal{H}_\phi \otimes \mathcal{H}_\psi, \mathcal{D}_\phi \otimes \mathcal{D}_\psi\} \quad (f \in E) \quad (3.12)$$

Proof See [11], Lemma II.4.8. ■

The proof of the following Proposition is straightforward.

Proposition 3.7

(a) Let ϕ and ψ be CCR states on ${}^2\mathcal{D}(M)$. Then the vector state $\Omega_\phi \otimes \Omega_\psi$ of the field

$$\{2^{-\frac{1}{2}}[\pi_\phi(f, g) \otimes 1_{\psi} + 1_{\phi} \otimes \pi_\psi(f, g)], \mathcal{H}_\phi \otimes \mathcal{H}_\psi, \mathcal{D}_\phi \otimes \mathcal{D}_\psi\} \quad (3.13)$$

$(f, g) \in {}^2\mathcal{D}(M)$, is a CCR state on ${}^2\mathcal{D}(M)$.

(b) Let ϕ, ψ be CAR states on ${}^2\mathcal{D}(M)$, and U_ψ a unitary operator on \mathcal{H}_ψ such that

$$\pi_\psi(f, 0)U_\psi + U_\psi \pi_\psi(f, 0) = 0 \quad \text{for all } f \in {}^2\mathcal{D}(M)$$

Then the vector state $\Omega_\phi \otimes \Omega_\psi$ of the field

$$\{2^{-\frac{1}{2}}[\pi_\phi(f,0) \otimes U_{\psi\phi} + \pi_\phi(0,g) \otimes U_{\psi\phi}^* + 1_\phi \otimes \pi_\psi(f,g)],$$

$$\mathcal{H}_\phi \otimes \mathcal{H}_\psi, \mathcal{D}_\phi \otimes \mathcal{D}_\psi\} \quad (3.14)$$

$(f,g) \in {}^2\mathcal{D}(M)$, is a CAR state on ${}^2\mathcal{D}(M)$. ■

The above definition of the s-product for CAR states is due to Mathon and Streater [27].

3.3 Representations of Current Algebra

We start our study of current algebras with the following definition due to Streater [28] and Araki [29].

Definition 3.8

(a) Let \mathcal{G} be a finite dimensional real Lie algebra and M a C^∞ -manifold countable at infinity. The current algebra on M based on \mathcal{G} is the space $\mathcal{D}_{\mathbb{R}}(M, \mathcal{G}) \simeq \mathcal{D}_{\mathbb{R}}(M) \otimes \mathcal{G}$ equipped with pointwise operations (multiplication by scalars, addition and Lie bracket).

(b) A cyclic representation of the current algebra $\mathcal{D}_{\mathbb{R}}(M, \mathcal{G})$ is a homomorphism $f \mapsto \pi(f)$ from $\mathcal{D}_{\mathbb{R}}(M, \mathcal{G})$ to anti-symmetric (unbounded) linear operators of a Hilbert space \mathcal{H} such that

(i) $\text{dom}[\pi(f)] \supset \mathcal{D}, \bar{\mathcal{D}} = \mathcal{H}$

(ii) $\pi(f)\mathcal{D} \subset \mathcal{D}$

(iii) there exists a cyclic vector $\Omega \in \mathcal{D}$

(iv) $f \mapsto \langle \Psi, \pi(f)\Phi \rangle$ is continuous for all $\Psi, \Phi \in \mathcal{D}$ and

(v) $[\pi(f), \pi(g)]\Phi = \pi([f, g])\Phi$, for all $f, g \in \mathcal{D}_{\mathbb{R}}(M, \mathcal{G})$, $\Phi \in \mathcal{D}$.

Let $M = \mathbb{R}^n$, let $\{X_i\}_{1 \leq i \leq d}$ be a basis for \mathcal{G} , and $\{C_{ij}^k\}_{1 \leq i, j, k \leq d}$

the structure constants of \mathcal{G} with respect to this basis. The formal unsmeared field operators $\pi_i(x)$, "defined" by

$$\pi(f) = \int \pi_i(x) f_i(x) dx, \quad f = f_i(x) \otimes X_i, \quad (3.15)$$

obey the relations

$$[\pi_i(x), \pi_j(y)] = C_{ij}^k \pi_k(x) \delta(x-y) \quad (3.16)$$

When $\mathcal{G} = \mathfrak{su}(2)$ we are interested in characterizing those cyclic representations for which the formal quantity (the Heisenberg Hamiltonian)

$$\iint \pi_i(x) J_{ij}(x-y) \pi_j(y) dx dy, \quad J_{ij}(x-y) = J_{ji}(y-x) \geq 0, \quad J_{ij}(x) \in \mathcal{D}(\mathbb{R}^n) \quad (3.17)$$

is a well defined operator. Note that since $J_{ij}(x-y) \notin \mathcal{D}(\mathbb{R}^{2n})$ the Heisenberg Hamiltonian is not defined in general, i.e., in every cyclic representation.

We now introduce the pertinent field algebra, $\underline{\mathcal{D}}(M, \mathcal{G})$, and its relation to the current commutation relations.

Definition 3.9

(a) $\underline{\mathcal{D}}(M, \mathcal{G})$ will be the BU-algebra based on $\mathcal{D}_{\mathbb{R}}(M, \mathcal{G})$ i.e.

$$\underline{\mathcal{D}}(M, \mathcal{G}) = \bigoplus_{n=0}^{\infty} \overline{\otimes}^n \mathcal{D}(M, \mathcal{G}), \quad \mathcal{D}(M, \mathcal{G}) = \mathcal{D}_{\mathbb{R}}(M, \mathcal{G}) \otimes_{\mathbb{R}} \mathbb{C} \quad (3.18)$$

and $[f(x) \otimes X]^* = -f(x) \otimes X$ if $f(x) \otimes X \in \mathcal{D}_{\mathbb{R}}(M, \mathcal{G})$

(b) A state on $\underline{\mathcal{D}}(M, \mathcal{G})$ which annihilates the smallest closed * -ideal generated by elements of the form

$$(0, [f, g], g \otimes f - f \otimes g, 0, \dots) \quad f, g \in \mathcal{D}_{\mathbb{R}}(M, \mathcal{G}) \quad (3.19)$$

will be called a spin state.

Proposition 3.10 A cyclic representation of $\mathcal{D}_{\mathbb{R}}(M, \mathcal{G})$ determines a spin state on $\underline{\mathcal{D}}(M, \mathcal{G})$ and conversely.

Proof This is a particular case of the Wightman reconstruction theorem [23] for BU-algebras. ■

We now consider the case $M = \mathbb{R}^n$, $\mathcal{G} = \mathfrak{su}(2)$ and use equation (3.16) to compute formally, the commutator

$$\left[\iint \pi_i(x) J_{ij}(x-y) \pi_j(y) dx dy, \int \pi_k(z) f_k(z) dz \right],$$

getting

$$\iint \pi_i(x) \{ J_{ij}(x-y) f_k(x) C_{lk}^i + J_{il}(x-y) f_k(y) C_{lk}^j \} \pi_j(y) dy \quad (3.20)$$

Note that now the quantity in braces is an element of $\mathcal{D}(\mathbb{R}^{2n})$. This results hints at the possibility of giving the Heisenberg Hamiltonian a rigorous meaning as a continuous derivation δ on $\mathcal{D}(\mathbb{R}^n, \mathcal{G})$. The next Proposition shows that this is indeed the case.

Proposition 3.11 The map $\delta : \mathcal{D}(\mathbb{R}^n, \mathcal{G}) \rightarrow \mathcal{D}(\mathbb{R}^n, \mathcal{G})$ given by:

$$\delta(1) = 0, \quad \delta(f) = (0, 0, \dots, f_{i_1 i_2 \dots i_{p+1}}(x_1, x_2, \dots, x_{p+1}) \otimes X_{i_1} \otimes X_{i_2} \otimes \dots \otimes X_{i_{p+1}}, 0, \dots)$$

for

$$f = (0, 0, \dots, f_{i_1 i_2 \dots i_p}(x_1, x_2, \dots, x_p) \otimes X_{i_1} \otimes X_{i_2} \otimes \dots \otimes X_{i_p}, 0, \dots) \quad (3.21)$$

where

$$f_{i_1 i_2 \dots i_{p+1}}(x_1, x_2, \dots, x_{p+1}) = J_{li_2}(x_1 - x_2) C_{lk}^{i_1} f_{ki_2 \dots i_{p+1}}(x_1, x_3, \dots, x_{p+1}) + J_{li_1}(x_1 - x_2) C_{lk}^{i_2} f_{ki_3 \dots i_{p+1}}(x_2, x_3, \dots, x_{p+1}) + \dots +$$

$$+J_{i_p \ell} (x_p - x_{p+1}) C_{\ell k}^{i_{p+1}} f_{i_1 i_2 \dots i_{p-1} k} (x_1, x_2, \dots, x_{p-1}, x_{p+1}) \quad (3.22)$$

is a $*$ -antisymmetric continuous outer derivation on $\mathcal{D}(\mathbb{R}^n, \mathcal{G})$. It is an infinitesimal generator iff for all $f \in \mathcal{D}(\mathbb{R}^n, \mathcal{G})$ there exists a natural number $n=n(f)$ such that $\delta^n(f) = 0$.

Proof First we remark that the proposed formula for δ is obtained by a repeated application of the formal manipulations that led to equation (3.20). The $*$ -antisymmetry of δ , $\delta(f)^* = -\delta(f^*)$ for all $f \in \mathcal{D}(\mathbb{R}^n, \mathcal{G})$, follows from the symmetry of the J_{ij} given in equation (3.17). For the continuity of δ , since $\mathcal{D}(\mathbb{R}^n, \mathcal{G})$ is an LF-space, it is enough to show that if $\{f_{i_1 i_2 \dots i_p}^{(m)}\}$ is a zero-convergent sequence in $\mathcal{D}(\mathbb{R}^{np})$, then $\{f_{i_1 i_2 \dots i_{p+1}}^{(m)}\}$ is a zero-convergent sequence in $\mathcal{D}(\mathbb{R}^{np+n})$: [13],

Proposition 14.7. Let $K_1 \times \dots \times K_p \subset \mathbb{R}^{np}$ be a compact set containing the supports of all the $\{f_{i_1 i_2 \dots i_p}^{(m)}\}$ and on which these functions and their

derivatives tend to zero uniformly as $m \rightarrow \infty$. It is clear that all the

$\{f_{i_1 i_2 \dots i_{p+1}}^{(m)}\}$ have their support in $K'_1 \times K'_2 \times \dots \times K'_{p+1} \subset \mathbb{R}^{np+n}$,

$K'_1 = K_1 \cup (K_1 + K)$, $K'_i = (K_{i-1} - K) \cup K_{i-1} \cup K_i \cup (K_i + K)$, $1 < i < p+1$, $K'_{p+1} = (K_p - K) \cup K_p$,

$K = \bigcup_{ij} \text{supp } J_{ij}$, and tend to zero uniformly these together with their derivatives.

The proof that δ is an outer derivation is straightforward. The last assertion is proven in Appendix C, Proposition C.9. ■

We will refer to this derivation as the Heisenberg derivation.

It can be shown that δ is an infinitesimal generator when \mathcal{G} is a

nilpotent Lie algebra, but not when $\mathcal{G} = \mathfrak{su}(2)$.

We remark that as a consequence of the existence of an outer derivation the first cohomology group $H^1(\underline{\mathcal{D}}(\mathbb{R}^n, \mathcal{G}), \underline{\mathcal{D}}(\mathbb{R}^n, \mathcal{G}))$ is non-trivial : [30]

The following Proposition characterizes the spin states on which the Heisenberg derivation is implemented by a (representation-dependent) Hamiltonian.

Proposition 3.12 Let ϕ be a spin state on $\underline{\mathcal{D}}(\mathbb{R}^n, \mathcal{G})$. Then there is a symmetric operator H_ϕ on \mathcal{H}_ϕ with $\mathcal{D}(H_\phi) = \mathcal{D}_\phi$ and satisfying

$$\pi_\phi(\delta(f))[g]_\phi = [H_\phi, \pi_\phi(f)][g]_\phi \text{ for all } f \in \underline{\mathcal{D}}(\mathbb{R}^n, \mathcal{G}), [g]_\phi \in \mathcal{D}_\phi$$

iff there exists a constant $L \geq 0$ such that

$$|\phi(\delta(f))|^2 \leq L[\phi(f^*f + ff^*)] \text{ for all } f \in \underline{\mathcal{D}}(\mathbb{R}^n, \mathcal{G}) \quad (3.23)$$

Proof As in [31], Theorem 3. ■

In section 5.2 we propose a dynamical scheme to characterize those stationary states, $\phi(\delta(f)) = 0$, that could be interpreted as states of thermal equilibrium, the so called KMS states.

In what remains of this section we give a procedure to construct representations of $\underline{\mathcal{D}}(\mathbb{R}^n, \mathcal{G})$ from certain representations of \mathcal{G} .

Let T be a representation of \mathcal{G} by first order linear differential operators, as when G , a local Lie group whose Lie algebra is isomorphic to \mathcal{G} , acts effectively as a local Lie transformation group on a manifold W of dimension m . There then exists a basis $\{X_i\}_{1 \leq i \leq d}$ for \mathcal{G} such that [32]

$$T(X_i) = \sum_{r=1}^m P_{ri}(x_1, \dots, x_m) \frac{\partial}{\partial x_r} \quad (3.24)$$

in local coordinates. We suppose further that the functions $P_{ri}(x_1, \dots, x_m)$, $1 \leq r \leq m$, $1 \leq i \leq d$, are entire and such that for all m -tuples (v_1, \dots, v_m) of non-negative integers and all $(f_1, \dots, f_m) \in \mathcal{D}(\mathbb{R}^n)^m$, the functions $\frac{\partial^{v_1 + \dots + v_m}}{\partial x_1^{v_1} \dots \partial x_m^{v_m}} P_{ri}(x_1, \dots, x_m)$, after the replacement $x_i \rightarrow f_i$, are multipliers for $\mathcal{D}(\mathbb{R}^n)$.

We shall consider a class of representations of $\mathcal{D}(\mathbb{R}^n, \mathcal{G})$ whose formal unsmeared field operators $\pi_i(x)$ (c.f. equation (3.15)) have the form

$$\pi_i(x) = \sum_{r=1}^m P_{ri}(F_1(x), \dots, F_m(x)) \frac{\delta}{\delta F_r(x)} \quad (3.25)$$

where $(F_1, \dots, F_m) \in \mathcal{D}(\mathbb{R}^n)^m$.

In order to give a rigorous meaning to the "functional derivative" in (3.25), let us recall the theory of Fréchet - Volterra derivatives as expounded in [33]. For E a complete lcs and Ω a connected open subset of \mathbb{R}^n , consider mappings $\Phi : \mathcal{D}(\Omega) \rightarrow E$ which are continuous and analytic in the sense that for every $f_1, f_2 \in \mathcal{D}(\Omega)$, the function $z \mapsto \Phi(f_1 + zf_2)$ is an E -valued entire function.

For every such Φ define the E -elements

$$\delta\Phi(f_1; f_2) = \left. \frac{d}{dz} \Phi(f_1 + zf_2) \right|_{z=0} \quad (f_1, f_2 \in \mathcal{D}(\Omega)) \quad (3.26)$$

For fixed $f_1, f_2 \mapsto \delta\Phi(f_1; f_2)$ defines a distribution, conventionally written $\delta\Phi(f)/\delta f$

$$\frac{\delta\Phi(f)}{\delta f}(g) = \delta\Phi(f; g) \quad (f, g \in \mathcal{D}(\Omega)). \quad (3.27)$$

Clearly $\delta\Phi(f)/\delta f \in \mathcal{D}'(\Omega; E)$, and is the FV-derivative of Φ .

Higher order derivatives are defined similarly. For all

$f, f_1, \dots, f_n \in \mathcal{D}(\Omega)$ set

$$\delta^n \Phi(f; f_1, \dots, f_n) = \frac{\partial^n}{\partial z_1 \dots \partial z_n} \Phi(f + z_1 f_1 + \dots + z_n f_n) \Big|_{z_i=0}, \quad (3.28)$$

and then

$$\frac{\delta^n \Phi(f)}{\delta f^n} (f_1, \dots, f_n) = \delta^n \Phi(f; f_1, \dots, f_n). \quad (3.29)$$

Thus $\delta^n \Phi(f)/\delta f^n \in \mathcal{D}'(\Omega^n; E)$:

The Volterra expansion of Φ is the convergent series

$$\Phi(f+zg) = \sum_{n+1 \in \mathbb{N}} \frac{z^n}{n!} \frac{\delta^n \Phi(f)}{\delta f^n} (g, g, \dots, g), \quad (3.30)$$

We need the multi-dimensional form of this theory. Let $F = (f_1, \dots, f_p) \in \mathcal{D}(\Omega)^p$, and suppose $\Phi : \mathcal{D}(\Omega)^p \rightarrow E$ is continuous and for every $F, F' \in \mathcal{D}(\Omega)^p$

$$(z_1, \dots, z_p) \mapsto \Phi(f_1 + z_1 f_1', \dots, f_p + z_p f_p')$$

is entire analytic from \mathbb{C}^p to E . For each $1 \leq j \leq p$,

$$\delta_j \Phi(F; F') = \frac{\partial}{\partial z_j} \Phi(f_1, \dots, f_j + z_j f_j', \dots, f_p) \Big|_{z_j=0} \quad (3.31)$$

defines $\delta\Phi/\delta f_j \in \mathcal{D}'(\Omega; E)$ by

$$\frac{\delta\Phi(F)}{\delta f_j} (f_j') = \delta_j \Phi(F; F') \quad (3.32)$$

In the same way we can construct $\delta^n \Phi(F)/\delta f_1^{m_1} \dots \delta f_p^{m_p}$, with $n = m_1 + \dots + m_p$; this is an element of $\mathcal{D}'(\Omega^n; E)$.

With these conventions and notations, let $T(\mathcal{G})$ be given as above and

$\Phi : \mathcal{D}(\mathbb{R}^n)^m \rightarrow \mathcal{H}$ where we assume $E = \mathcal{H}$ to be a Hilbert space. Recall

that $m = \dim(W)$. We define a family $\{\pi(f) : f \in \mathcal{D}(\mathbb{R}^n, \mathcal{G})\}$ of (unbounded) operators on \mathcal{H} by

$$\pi(f)\phi(F) = \sum_{r=1}^m \frac{\delta\phi(F)}{\delta F_r} (g_r(f, F)) \quad (3.33)$$

where $g_r(f, F) = \sum_{i=1}^d f_i P_{ri}(F_1, \dots, F_m)$ is in $\mathcal{D}(\Omega)$, $f = (f_1, \dots, f_d) \in \mathcal{D}_{\mathbb{R}}(\mathbb{R}^n, \mathcal{G})$ with respect to the basis $\{X_i\}_{1 \leq i \leq d}$ and $F = (F_1, \dots, F_m) \in \mathcal{D}(\mathbb{R}^n)^m$.

It is straightforward to show that for any pair $f, g \in \mathcal{D}(\mathbb{R}^n, \mathcal{G})$, and $F \in \mathcal{D}(\mathbb{R}^n)^m$

$$[\pi(f), \pi(g)]\phi(F) = \pi([f, g])\phi(F). \quad (3.34)$$

In order that the $\{\pi(f) : f \in \mathcal{D}_{\mathbb{R}}(\mathbb{R}^n, \mathcal{G})\}$ have a common dense domain we now suppose that the linear span $\mathcal{D} = \mathbf{V}\{\phi(F) : F \in \mathcal{D}(\mathbb{R}^n)^m\}$ is dense in \mathcal{H} . For by the properties of this set, \mathcal{D} is stable under the $\pi(f)$. The operators $\{\pi(f) : f \in \mathcal{D}_{\mathbb{R}}(\mathbb{R}^n, \mathcal{G})\}$ will not be antisymmetric in general. The fulfilment of this condition, must be considered separately in each case : we have done so for $su(2)$.

For $su(2)$ we proceed as follows. We take $W = \mathbb{R}^3$, $T(X_i) = \epsilon_{ijk} x_j \partial / \partial x_k$, and for \mathcal{H} the symmetric Fock space constructed from $L^2(\mathbb{R}^n, dx) \oplus L^2(\mathbb{R}^n, dx) \oplus L^2(\mathbb{R}^n, dx)$. Let $\text{Ent}(\mathcal{D}(\mathbb{R}^n)^3, \mathcal{H})$ be the set of functions $\phi : \mathcal{D}(\mathbb{R}^n)^3 \rightarrow \mathcal{H}$ which are continuous and entire analytic as above; and let $\text{Exp}(\mathcal{D}(\mathbb{R}^n)^3)$ be the set of coherent vectors, i.e., those of the form

$$E(F) = (1, F, \dots, (n!)^{-\frac{1}{2}} F^{\otimes n}, \dots), \quad (F \in \mathcal{D}(\mathbb{R}^n)^3)$$

It is well known that $F \mapsto E(F)$ is continuous and Exp is a total set of \mathcal{H} , see [34]. It is perhaps not so well known that they have the requisite analyticity properties. But

$$\left\| \frac{\partial}{\partial z_i} E(F + z_1 F'_1 + z_2 F'_2 + z_3 F'_3) \right\|^2 \leq$$

$$\leq ||F_i^2||^2 \sum_{n \in \mathbb{N}} (n!)^{-1} n^2 ||F+z_1 F_1' + z_2 F_2' + z_3 F_3' ||^{2n-2} < +\infty$$

so it has complex derivatives everywhere. The action of $\pi(f)$ on Exp is given by

$$\pi(f)E(F) = (0, \sigma(f)F, (2!)^{-\frac{1}{2}}[\sigma(f)F \otimes F + F \otimes \sigma(f)F], \dots) \quad (3.35)$$

where (summation convention)

$$(\sigma(f)F)_i = \epsilon_{ijk} f_j F_k \quad (1 \leq i, j, k \leq 3) \quad (3.36)$$

for $f \in \mathcal{D}_{\mathbb{R}}(\mathbb{R}^n, \mathcal{G})$, $F \in \mathcal{D}(\mathbb{R}^n)^3$.

It is not difficult to show that

$$||\pi(f)E(F)|| = ||\sigma(f)F|| + ||E(F)|| \quad (3.37)$$

and

$$\langle \pi(f)E(F), E(F') \rangle = -\langle E(F), \pi(f)E(F') \rangle \quad (f \in \mathcal{D}_{\mathbb{R}}(\mathbb{R}^n, \mathcal{G})) \quad (3.38)$$

so the operators $\{\pi(f) : f \in \mathcal{D}_{\mathbb{R}}(\mathbb{R}^n, \mathcal{G})\}$ are unbounded and antisymmetric.

Now a simple computation gives

$$\begin{aligned} & ||\pi(f_1) \dots \pi(f_r)E(F)||^2 \\ & \leq ||\pi(f_1) \dots \pi(f_r) \sum_{k=0}^{r-1} (k!)^{-\frac{1}{2}} F^{\otimes k} ||^2 + \sum_{k=r}^{\infty} (k!)^{-1} k^{2r} \mu^2 ||F||^{2k-2r} < +\infty \end{aligned}$$

where μ is the maximum norm of the r^r vectors in the expansion of

$\pi(f_1) \dots \pi(f_r) F^{\otimes r}$ as product vectors $F_1 \otimes \dots \otimes F_r$.

Thus the linear span

$$\mathcal{D} = \bigvee \{E(F), \pi(f_1) \dots \pi(f_r)E(F) : f_1, \dots, f_r \in \mathcal{D}_{\mathbb{R}}(\mathbb{R}^n, \mathcal{G}); F \in \mathcal{D}(\mathbb{R}^n)^3; r \geq 1\}$$

is a common dense domain for the fields $\{\pi(f)\}$.

Every vector state of the above field $\{\pi(f), \mathcal{H}, \mathcal{D}\}$ associated with a (normalized) vector \mathcal{Q} is a spin state on $\underline{\mathcal{D}}(\mathbb{R}^n, \mathcal{G})$.

We recall that a state ϕ on a BU-algebra E is infinitely divisible if for each natural number n , there is a state ω_n such that $\phi = (\omega_n)^{\otimes n}$, c.f. Definition 3.5. A state is a character if $\phi|_{\overline{\otimes}^n E} = T^{\otimes n}$, for all $n \geq 1$, where $T \in E_{\mathbb{R}}$. A state is prime, if it is not a character and cannot be decomposed into the s-product of other states, except trivially. The following factorization theorem is due to Hegerfeldt [35].

Proposition 3.13 Every state on a BU-algebra is an s-product of two states, one of which is either 1 or the s-product of an at most denumerable number of prime states; and the other is infinitely divisible with no prime factors. ■

We now give an example of a spin state on $\mathcal{D}(\mathbb{R}^n, \mathcal{G})$ with a prime part. We take $W = \mathbb{R}^3$, $T(X_i) = \varepsilon_{ijk} x_j \partial / \partial x_k$, $\mathcal{H} = L^2(\mathbb{R}^n, dx) \oplus L^2(\mathbb{R}^n, dx) \oplus L^2(\mathbb{R}^n, dx)$ and $\mathcal{A}(F) = F$. This gives the fields operators $\{\sigma(f) : f \in \mathcal{D}_{\mathbb{R}}(\mathbb{R}^n, \mathcal{G})\}$ introduced above:

$$(\sigma(f)F)_i(\underline{x}) = \varepsilon_{ijk} f_j(\underline{x}) F_k(\underline{x}) \quad (3.39)$$

These operators are bounded

$$\|\sigma(f)\| = \max_{\underline{x} \in \mathbb{R}^n} [f_1(\underline{x})^2 + f_2(\underline{x})^2 + f_3(\underline{x})^2]^{\frac{1}{2}} \quad (3.40)$$

consider the vector state $\Omega = f \otimes X_1$, $\|\Omega\| = \|f\|_2 = 1$, $f \in \mathcal{D}_{\mathbb{R}}(\mathbb{R}^n)$, of the field $\{\sigma(f), \mathcal{H}\}$.

To see that $(\sigma, \mathcal{D}_{\Omega}, \Omega)$ has a prime part, let $h_i \in \mathcal{D}_{\mathbb{R}}(\mathbb{R}^n)$ ($1 \leq i \leq 4$) and let $h = (h_1, h_2, h_3)$, $h' = (h_4, -h_3, h_2)$ be elements of $\mathcal{D}_{\mathbb{R}}(\mathbb{R}^n, \mathcal{G})$. A simple calculation yields

$$\begin{aligned} & \langle \Omega, \sigma(h')^* \sigma(h)^* \sigma(h) \sigma(h') \Omega \rangle_T \\ &= \int_{\mathbb{R}^n} f^2(\underline{x}) h_1^2(\underline{x}) [h_2^2(\underline{x}) + h_3^2(\underline{x})] d\underline{x} - 4 \left\{ \int_{\mathbb{R}^n} f^2(\underline{x}) [h_2^2(\underline{x}) + h_3^2(\underline{x})] d\underline{x} \right\}^2 \end{aligned} \quad (3.41)$$

where $\langle \Omega, \cdot \rangle_T$ is the truncated functional associated with the vector state $\langle \Omega, \cdot \rangle$. Choosing h_1 , such that $\text{supp}(h_1) \cap \text{supp}(f) = \emptyset$ and h_2 such that $\text{supp}(h_2) \cap \text{supp}(f) \neq \emptyset$ we get

$$\langle \Omega, \sigma(h'_1)^* \sigma(h)^* \sigma(h) \sigma(h'_2) \rangle_T < 0 \quad (3.42)$$

by [35], Theorem 2.1, $(\sigma, \mathcal{D}_\Omega, \Omega)$ has a prime part.

CHAPTER 4

SYMMETRIES

4.1 Definitions and Preliminaries

We pattern our discussion of symmetries after the corresponding theory for C^* -algebras : [24], [36].

Definition 4.1 Let \mathcal{A} be an I^* -algebra and $\text{Aut}(\mathcal{A})$ the group of its automorphisms. We call I^* -algebra with a group of automorphisms a triple (\mathcal{A}, G, α) where G is a group and $\alpha : G \longrightarrow \text{Aut}(\mathcal{A})$ is a (group) homomorphism. If $g \in G$ we denote by α_g the corresponding automorphism of \mathcal{A} .

When G is a topological group we say that α is continuous if the functions $g \longmapsto \alpha_g(f)$, from G to $\mathcal{A}[\mathcal{I}]$, are continuous for all $f \in \mathcal{A}$.

It can be shown that if \mathcal{A} has property (N), α is continuous iff the functions $g \longmapsto \phi[\alpha_g(f)]$ are continuous for all $\phi \in \mathcal{E}(\mathcal{A})$, $f \in \mathcal{A}$.

In view of the applications we have in mind, from now on G will be taken to be locally compact, non-compact, amenable and second countable.

Definition 4.2 A state ϕ on \mathcal{A} is G -invariant if $\phi \circ \alpha_g = \phi$ for all $g \in G$. The extreme points of the set of G -invariant states are termed G -ergodic states.

For C^* -algebras the existence of G -invariant states is derived from the fact that $g \longmapsto \phi[\alpha_g(f)]$ is in $\text{CB}(G)$ for all $\phi \in \mathcal{E}(\mathcal{A})$, $f \in \mathcal{A}$: [24], Lemma (p.172). For I^* -algebras the corresponding functions are in $C(G)$, and it is an open problem whether or not there exists a left invariant

mean on this space : [37].

A sufficient condition for the construction of G -invariant states has been given by Hofmann and Lassner [38] : Let K_G be the closed linear subspace generated by elements of the form $f - \alpha_g(f)$, $f \in \mathcal{A}$, $g \in G$. Since $f \cdot^* c K_G$ if $f \in K_G$, the linear space K_G has the decomposition

$$K_G = L_G + iL_G$$

where L_G is a real subspace of \mathcal{A}_h . Then if T is real linear on a subspace of \mathcal{A}_h satisfying either of the conditions

$$\begin{aligned} \mathcal{A}_h &\neq \text{cl}(\overline{\mathcal{A}_+} + L_G + \ker T) \\ -1 &\notin \text{cl}(\overline{\mathcal{A}_+} + L_G + \ker T) \end{aligned}$$

it has a positive G -invariant extension to \mathcal{A} (the closures being taken with respect to the topology \mathcal{T}).

Theorem 4.3 The group $\alpha(G)$ is unitarily implemented in every G -invariant state ϕ : there exists a strongly continuous unitary representation $g \mapsto \mathcal{U}_g^\phi$ of G on \mathcal{H}_ϕ such that \mathcal{D}_ϕ is stable, $\mathcal{U}_g^\phi \mathcal{D}_\phi = \mathcal{D}_\phi$ for all $g \in G$; the vacuum is invariant, $\mathcal{U}_g^\phi \Omega_\phi = \Omega_\phi$ for all $g \in G$; and

$$\pi_\phi(\alpha_g(f))^\phi = \mathcal{U}_g^\phi \pi_\phi(f) (\mathcal{U}_g^\phi)^* \quad (\phi \in \mathcal{D}_\phi, f \in \mathcal{A}, g \in G)$$

Conversely, a cyclic representation of \mathcal{A} as a $*$ -operator family with $\alpha(G)$ unitarily implemented as above leads to a G -invariant state on \mathcal{A} .

Proof As in [11], Theorem I.5.2. ■

The following Definition and Lemma are due to Dubin.

Definition 4.4 A state ϕ is G-weakly asymptotically abelian, or waa, if for every pair $\phi, \psi \in \mathcal{D}_\phi$, every pair $f, h \in \mathcal{A}$, and every $\epsilon > 0$ there exists a compact $\Delta \subset G$ such that

$$|\langle \phi, [\pi_\phi(\alpha_g(f)), \pi_\phi(h)] \psi \rangle| < \epsilon$$

for all $g \in G \setminus \Delta$.

Lemma 4.5 For a waa state ϕ on \mathcal{A} , the function

$$k(g) = ||\pi_\phi(f) \mathcal{U}_g^{\phi_\Phi}||$$

is in $CB(G)$, for every $f \in \mathcal{A}$ and $\Phi \in \mathcal{D}_\phi$.

Proof Since $\mathcal{A}[\mathcal{I}]$ is barrelled and π_ϕ is weakly continuous, it is also strongly continuous ([21], Theorem 4.1), i.e., the map

$f \mapsto ||\pi_\phi(f) \Phi||$ is continuous for all $\Phi \in \mathcal{D}_\phi$. The continuity of $k(g)$ now follows from the fact that it is the composition of the continuous maps $g \mapsto \alpha_{-1}^g(f)$, and

$$\alpha_{-1}^g(f) \mapsto ||\pi_\phi(\alpha_{-1}^g(f)) \Phi|| = ||\pi_\phi(f) \mathcal{U}_g^{\phi_\Phi}||.$$

It remains to prove that $k(g)$ is bounded. Clearly we only need to check this property on the complement of a compact set, since $k(g)$ is continuous. Now

$$k(g)^2 = \langle \phi, \pi_\phi(\alpha_{-1}^g(f^* f)) \pi_\phi(h) \Omega_\phi \rangle \leq$$

$$|\langle \phi, [\pi_\phi(\alpha_{-1}^g(f^* f)) \pi_\phi(h)] \Omega_\phi \rangle| + |\langle \phi, \pi_\phi(h) \pi_\phi(\alpha_{-1}^g(f^* f)) \Omega_\phi \rangle|$$

where $\Phi = [h]_\phi$. By waa the first term can be made as small as we want in the complement of a compact set and the second is bounded, for all g , by $||\pi_\phi(h^*) \Phi|| ||\pi_\phi(f^* f) \Omega_\phi||$. Therefore $k(g)$ is bounded. ■

4.2 Ergodic Theory

Let P be the orthogonal projection from \mathcal{H}_ϕ onto the G-invariant vectors.

We shall show that P behaves well with regard to operator domains.

Proposition 4.6 For a G -invariant ^{waa}state ϕ , we have

$$P(\mathcal{D}_\phi) \subset \mathcal{D}_\phi^{**}$$

where $\mathcal{D}_\phi^{**} = \bigcap_{f \in A} \text{dom}[\pi_\phi(f)^{**}]$

Proof First we note that Størmer [39], Theorem 2.2, has shown that there exists a net $\{A_\nu \in \Gamma \mathcal{U}^\phi(G) : \nu \in I\}$ in the convex hull of $\mathcal{U}^\phi(G)$, which converges to P in the strong operator topology. Now as \mathcal{H}_ϕ is separable and $\Gamma \mathcal{U}^\phi(G)$ is a bounded subset of $B(\mathcal{H}_\phi)$ in the uniform operator topology, the strong operator topology may be described by a norm : [40], Proposition 2.3.2. Therefore the above net contains a subsequence $\{A_n : n=1,2,\dots\}$ converging to P in the strong operator topology.

Taking $A \in \Gamma \mathcal{U}^\phi(G)$, it is clear from the previous lemma that

$\|\pi_\phi(f)A\phi\| \leq M/2$ where the constant M depends upon $f \in A$, $\phi \in \mathcal{D}_\phi$, but not on A . Then for the above sequence,

$$\|\pi_\phi(f)(A_n - A_m)\phi\| \leq M$$

We now show that if $\psi \in \text{dom}[\pi_\phi(f)^*]$, then

$$\lim_{n,m \rightarrow \infty} \langle \psi, \pi_\phi(f)(A_n - A_m)\phi \rangle = 0$$

Evidently

$$\begin{aligned} \lim_{n,m \rightarrow \infty} \langle \psi, \pi_\phi(f)(A_n - A_m)\phi \rangle &= \lim_{n,m \rightarrow \infty} \langle \pi_\phi(f)^* \psi, (A_n - A_m)\phi \rangle \\ &= \langle \pi_\phi(f)^* \psi, (P-P)\phi \rangle = 0 \end{aligned}$$

The next step is to show that for all $\theta \in \mathcal{H}_\phi$, the same thing holds,

i.e. $\lim_{n,m \rightarrow \infty} \langle \theta, \pi_\phi(f)(A_n - A_m)\phi \rangle = 0$. Since $\pi_\phi(f)$ is densely defined and

closeable $\text{dom}[\pi_\phi(f)^*]$ is dense in \mathcal{H}_ϕ . Therefore for all $\theta \in \mathcal{H}_\phi$ and $\epsilon > 0$ there is $\Psi \in \text{dom}[\pi_\phi(f)^*]$ such that $||\theta - \Psi|| < \epsilon$. Then

$$\begin{aligned} ||\theta, \pi_\phi(f)(A_n - A_m)\Phi|| &\leq |<\theta - \Psi, \pi_\phi(f)(A_n - A_m)\Phi>| + |<\Psi, \pi_\phi(f)(A_n - A_m)\Phi>| \\ &\leq ||\theta - \Psi|| ||\pi_\phi(f)(A_n - A_m)\Phi|| + ||\Psi, \pi_\phi(f)(A_n - A_m)\Phi>| \\ &< M\epsilon + |<\Psi, \pi_\phi(f)(A_n - A_m)\Phi>| \end{aligned}$$

from which the desired conclusion follows because the second term in the last line can be made as small as we want by taking n and m sufficiently large. The sequence $\{\pi_\phi(f)A_n\Phi : n=1,2,\dots\}$ is therefore weakly convergent and as every Hilbert space is sequentially weakly complete ([41], p.186), there is a vector $\Xi \in \mathcal{H}_\phi$ such that $\lim_{n \rightarrow \infty} <\theta, \pi_\phi(f)A_n\Phi> = <\theta, \Xi>$ for all $\theta \in \mathcal{H}_\phi$.

Taking $\Psi \in \text{dom}[\pi_\phi(f)^*]$, it is clear from this that

$$<\Psi, \Xi> = <\pi_\phi(f)^*\Psi, P\Phi>$$

Then $P\Phi \in \text{dom}[\pi_\phi(f)^{**}]$ and we are done. ■

The following is straightforward.

Lemma 4.7

- (a) For any G -invariant state the domains \mathcal{D}_ϕ^* and \mathcal{D}_ϕ^{**} are all stable under $\mathcal{U}^\phi(G)$ and

$$\mathcal{U}_g^\phi \pi_\phi^\#(f) (\mathcal{U}_g^\phi)^* \Phi = \pi_\phi^\#(\alpha_g(f)) \Phi$$

for all $f \in \mathcal{A}$, $\Phi \in \mathcal{D}_\phi^\#$, and $g \in G$. Here $\pi_\phi^\#$ is π_ϕ^* or π_ϕ^{**} , and correspondingly for $\mathcal{D}_\phi^\#$. c.f. [42,43].

- (b) If, in addition, ϕ is waa, then for every $f, h \in \mathcal{A}$, $\Phi, \Psi \in \mathcal{D}_\phi^{**}$, and $\epsilon > 0$, there exists a compact set $\Delta \subset G$ such that

$$|\langle \Phi, [\pi_{\phi}^{**}(\alpha_g(f)), \pi_{\phi}^{**}(h)] \Psi \rangle| < \epsilon$$

for all $g \in G \setminus \Delta$. ■

The next proposition is an improvement of an earlier result of Dubin.

Proposition 4.8 For a G -invariant waa-state ϕ , the reduced family $\{P\pi_{\phi}^{**}(f)P : f \in \mathcal{A}\}$ is a strongly abelian $*$ -operator family with domain \mathcal{D}_{ϕ}^{**} .

Proof In the same way as we proved that $P(\mathcal{D}_{\phi}) \subset \mathcal{D}_{\phi}^{**}$, we can show that $P(\mathcal{D}_{\phi}^{**}) \subset \mathcal{D}_{\phi}^{**}$, since $\mathcal{D}_{\phi}^{****} = \mathcal{D}_{\phi}^{**}$ ([43]). Consider

$$k_1(g) = \langle \pi_{\phi}^{**}(f)^* P\Phi, \mathcal{U}_g^{\phi*} \pi_{\phi}^{**}(h) P\Psi \rangle$$

$$k_2(g) = \langle \pi_{\phi}^{**}(h)^* P\Phi, \mathcal{U}_g^{\phi} \pi_{\phi}^{**}(f) P\Psi \rangle$$

for $f, h \in \mathcal{A}$, $\Phi \in \mathcal{D}_{\phi}^*$ and $\Psi \in \mathcal{D}_{\phi}^{**}$. For ϕ waa, $k_1, k_2 \in CB(G)$ and their difference can be made as small as we want in the complement of a compact set. As G is non-compact the invariant mean vanishes on all continuous functions with compact support ([44], p.178). Therefore applying the invariant mean η to $k_1 - k_2$ we get zero. Using the mean ergodic theorem, [24], p.177,

$$0 = \eta(k_1) - \eta(k_2) = \langle \Phi, [P\pi_{\phi}^{**}(f)P, P\pi_{\phi}^{**}(h)P] \Psi \rangle$$

As \mathcal{D}_{ϕ}^* is dense in \mathcal{H}_{ϕ} , the result follows. ■

The next ergodic Theorem is partly a simplification of a result first proven by Dubin.

Theorem 4.9 Let ϕ be a G -invariant ω -state on an I^* -algebra \mathcal{A} .

Consider the following conditions

(i) ϕ is G -ergodic

(ii) Let $\pi_\phi(\mathcal{A}) \cup \mathcal{U}^\phi(G) = \mathcal{B}_\phi$; then $(\mathcal{B}_\phi, \mathcal{D}_\phi)'_w \simeq \mathbb{C}$.

(iii) The range of P is one dimensional, spanned by Ω_ϕ .

Then (i) \iff (ii) \iff (iii)

Proof First we prove that (i) \iff (ii).

Assume $(\mathcal{B}_\phi, \mathcal{D}_\phi)'_w \not\simeq \mathbb{C}$. Then as in [42], Theorem 6.3, we can show

that there exists $L \in (\mathcal{B}_\phi, \mathcal{D}_\phi)'_w$, such that $0 \leq L \leq I$, $L \notin \{\lambda I : \lambda \in \mathbb{C}\}$ and

$\langle \Omega_\phi, L \Omega_\phi \rangle > 0$. Now

$$\phi_1(f) = \langle \Omega_\phi, L \Omega_\phi \rangle^{-1} \langle \Omega_\phi, \pi_\phi(f) L \Omega_\phi \rangle$$

and

$$\phi_2(f) = \langle \Omega_\phi, (I-L) \Omega_\phi \rangle^{-1} \langle \Omega_\phi, \pi_\phi(f) (I-L) \Omega_\phi \rangle$$

are G -invariant states such that $\phi_1 \neq \phi_2$ and $\phi = \lambda \phi_1 + (1-\lambda) \phi_2$ where

$\lambda = \langle \Omega_\phi, L \Omega_\phi \rangle$. Therefore (i) \implies (ii).

Suppose now that $\phi = \lambda \phi_1 + (1-\lambda) \phi_2$, where $0 < \lambda < 1$ and $\phi_1 \neq \phi_2$ are G -invariant states. Then there exists (c.f. ibid) $L \in (\mathcal{B}_\phi, \mathcal{D}_\phi)'_w$, such that

$\lambda \phi_1(f^* h) = \langle \pi_\phi(f) \Omega_\phi, L \pi_\phi(h) \Omega_\phi \rangle$, $0 < L < \lambda I$, and $L \notin \{\mu I : \mu \in \mathbb{C}\}$. Therefore

(ii) \implies (i).

We prove next that for any G -invariant state, $(\mathcal{B}_\phi, \mathcal{D}_\phi)'_w$ equals

$(\pi_\phi(\mathcal{A}) \cup P, \mathcal{D}_\phi)'_w$. For this it suffices to show that for any

$L \in \pi_\phi(\mathcal{A})'_w$, $[L, \mathcal{U}^\phi(G)] = 0$ implies $[L, P] = 0$ and conversely. Let

$A_n \in \Gamma \mathcal{U}^\phi(G)$ be the sequence converging strongly to P as in Proposition 4.6

Then for all $\Phi, \Psi \in \mathcal{H}_\phi$,

$$\langle L^* \Phi, A_n \Psi \rangle = \langle \Phi, A_n L \Psi \rangle$$

Passing to the limit gives $[L, P] = 0$. Going the other way we use twice the relation $L\pi_\phi(f)\Omega_\phi = \pi_\phi^*(f)L\Omega_\phi$ ([42], Lemma 4.5) to get the chain

$$\begin{aligned} L\mathcal{U}_g^\phi \pi_\phi(f)\Omega_\phi &= L\pi_\phi(\alpha_g(f))\Omega_\phi = \pi_\phi^*(\alpha_g(f))L\Omega_\phi = \pi_\phi^*(\alpha_g(f))PL\Omega_\phi = \\ \mathcal{U}_g^\phi \pi_\phi^*(f)PL\Omega_\phi &= \mathcal{U}_g^\phi \pi_\phi^*(f)L\Omega_\phi = \mathcal{U}_g^\phi L\pi_\phi(f)\Omega_\phi. \end{aligned}$$

Therefore $[L, \mathcal{U}_g^\phi] = 0$ since \mathcal{D}_ϕ is dense.

Now let $L \in (\pi_\phi(A) \vee P, \mathcal{D}_\phi)_w'$ and $\Phi \in \mathcal{D}_\phi$. Then if P is one dimensional

$$\langle L\Omega_\phi, \Phi \rangle = \langle L\Omega_\phi, P\Phi \rangle = \langle L\Omega_\phi, \Omega_\phi \rangle \langle \Omega_\phi, \Phi \rangle$$

or $L\Omega_\phi = \lambda\Omega_\phi$, where $\lambda = \langle L\Omega_\phi, \Omega_\phi \rangle$. To finish the proof note that

$$\langle L[f]_\phi, [h]_\phi \rangle = \langle L\Omega_\phi[f^*h]_\phi \rangle = \overline{\lambda} \langle [f]_\phi, [h]_\phi \rangle$$

and therefore $L = \lambda I$. This shows that (iii) \implies (ii). ■

The problem of determining when (i) \implies (iii) is rather delicate in general. Although under the additional hypothesis that $P\pi_\phi^{**}(A)P$ is topologically semisimple, $P\pi_\phi^{**}(A)P$ is ρ -isomorphic to a subalgebra of $C(K)$ (K the characters), this is not sufficient [45]. We cannot conclude that this isomorphism implies that ϕ ergodic determines a character on the indicated subalgebra. For material on the uniqueness of the vacuum see [46].

For the sake of completeness we mention that there is an integral decomposition theory for states on an I^* -algebra.

Proposition 4.10 Let ϕ be a state on an I^* -algebra A . Then there exists a standard measure space Λ , a weakly measurable map $\lambda \mapsto \phi_\lambda$ from Λ to the extremal states on A and a positive measure μ on Λ with $\mu(\Lambda) = 1$ such that

$$(i) \quad \phi = \int_{\Lambda}^{\oplus} \phi_{\lambda} d\mu(\lambda)$$

$$(ii) \quad \pi_{\phi}(f)\Omega_{\phi} = \int_{\Lambda}^{\oplus} \pi_{\phi_{\lambda}}(f)\Omega_{\phi_{\lambda}} d\mu(\lambda)$$

(iii) For the left kernels we have $L(\phi) \subset L(\phi_{\lambda})$ μ -a.e.

(iv) If ϕ is G -invariant the $\{\phi_{\lambda}\}$ can be taken as G -ergodic μ -a.e. and the implementing unitary groups satisfy

$$\mathcal{U}_g^{\phi} = \int_{\Lambda}^{\oplus} \mathcal{U}_g^{\phi_{\lambda}} d\mu(\lambda)$$

Proof See [46], Theorem 3.10; [47]; and [48], Theorem 5.1. ■

4.3 Symmetries in BU-Algebras

The next Proposition gives a standard method to generate symmetries for BU-algebras.

Proposition 4.11 Let $\alpha(G) \subset L(E)$ be a continuous representation of G on E . Then α induces a continuous representation $\underline{\alpha}(G) \subset L(\underline{E})$.

Proof As G is second countable, locally compact, and Hausdorff, it is metrizable. We shall show that if $\alpha^i(G)$ is a continuous representation on E_i (barrelled; $i=1,2$), then $(\alpha^1 \otimes \alpha^2)(G)$ is a continuous representation on $E_1 \otimes_i E_2$. By Theorem A.47 of Appendix A, $(\alpha^1 \otimes \alpha^2)(G) \subset L(E_1 \otimes_i E_2)$. As $E_1 \otimes_i E_2$ is barrelled ([49], Chapter I, p.78), the representation $(\alpha^1 \otimes \alpha^2)(G)$ is continuous if it is weakly continuous ([50], Theorem 5, p.25), i.e.

$$g \rightsquigarrow \omega(\alpha_g^1(f_1) \otimes \alpha_g^2(f_2)) \quad (f_i \in E_i)$$

is continuous for all $\omega \in (E_1 \otimes_i E_2)'$. By definition of the τ -topology

$$\omega(\alpha_g^1(f_1), \alpha_g^2(f_2)) = \omega(\alpha_g^1(f_1) \otimes \alpha_g^2(f_2))$$

is a separately continuous bilinear form on $E_1 \times E_2$, so jointly continuous on bounded subsets of $E_1 \times E_2$ (c.f. Proposition 1.3). By the

metrizable of G we check sequential continuity : let $g_n \rightarrow g$,

so $\alpha_{g_n}^i(f_i)$ converges to $\alpha_g^i(f_i)$ and the sets $\{\alpha_{g_n}^i(f_i) : g_n\}$ are bounded.

Hence $\tilde{\omega}(\alpha_{g_n}^1(f_1), \alpha_{g_n}^2(f_2)) \rightarrow \tilde{\omega}(\alpha_g^1(f_1), \alpha_g^2(f_2))$.

Having shown the continuity of the representation $\alpha^1 \otimes \alpha^2$ we consider its extension $\alpha^1 \bar{\otimes} \alpha^2$ to $E_1 \bar{\otimes} E_2$. By Theorem A.47 of Appendix A,

$(\alpha^1 \bar{\otimes} \alpha^2)(G) \subset L(E_1 \bar{\otimes} E_2)$. Since $E_1 \bar{\otimes} E_2$ is barrelled, G locally compact and $(\alpha^1 \bar{\otimes} \alpha^2)(G)$ a continuous representation, $\{\alpha_g^1 \bar{\otimes} \alpha_g^2 : g \in K\}$

is equicontinuous for all compact $K \subset G$: [50], Lemma 4, p.24. By Lemma 2,

p.23 (ibid) $\{\alpha_g^1 \bar{\otimes} \alpha_g^2 : g \in K\}$ is also equicontinuous and therefore the

representation $\alpha^1 \bar{\otimes} \alpha^2$ is continuous: ibid Lemma 1, p.22. Now if

$\underline{\alpha}_g = \bigoplus_{n=0}^{\infty} \bar{\otimes}^n \alpha_g$, evidently, $\underline{\alpha}(G) \subset L(\underline{E})$ and $\underline{\alpha}(G)$ is a continuous representation. ■

We now turn to the field algebra ${}^2\mathcal{D}(\mathbb{R}^n)$ and the specific symmetries of space translations and gauge invariance.

Definition 4.12

- (a) The group of automorphisms ${}^2\sigma(\mathbb{R}^n) = \{ {}^2\sigma_a : a \in \mathbb{R}^n \}$, where ${}^2\sigma_a(f, g) = (f_a, g_a)$, $f_a(x) = f(x-a)$, $g_a(x) = g(x-a)$, $(f, g) \in {}^2\mathcal{D}(\mathbb{R}^n)$, will be termed the group of space translations on ${}^2\mathcal{D}(\mathbb{R}^n)$.
- (b) The group of automorphisms ${}^2\gamma(\mathcal{U}(1)) = \{ {}^2\gamma_\theta : 0 \leq \theta < 2\pi \}$, where ${}^2\gamma_\theta(f, g) = (e^{i\theta} f, e^{-i\theta} g)$, $(f, g) \in {}^2\mathcal{D}(\mathbb{R}^n)$, will be termed the group of gauge transformations on ${}^2\mathcal{D}(\mathbb{R}^n)$.

It is not difficult to prove that ${}^2_{\sigma}(\mathbb{R}^n)$ and ${}^2_{\gamma}(\mathcal{U}(1))$ are continuous representations of the groups \mathbb{R}^n and $\mathcal{U}(1)$, respectively. By Proposition 4.11, they give rise to continuous representations ${}^2_{\underline{\sigma}}(\mathbb{R}^n)$ and ${}^2_{\underline{\gamma}}(\mathcal{U}(1))$ on ${}^2_{\underline{\mathcal{D}}}(\mathbb{R}^n)$, with infinitesimal generators

$${}^2_{\underline{\partial}_i} = (0, \partial_i \oplus \partial_i, (\partial_i \oplus \partial_i) \otimes I + I \otimes (\partial_i \oplus \partial_i), \dots) \quad (1 \leq i \leq n)$$

and ${}^2_{\underline{N}} = (0, 1 \oplus -1, (1 \oplus -1) \otimes I + I \otimes (1 \oplus -1), \dots)$

which are continuous $*$ -derivations. The ${}^2_{\underline{\partial}_i}$ derivations are implemented by essentially self-adjoint operators on the translation invariant states : c.f. Proposition 3.12 and Theorem 4.3. The analogous statement for ${}^2_{\underline{N}}$ and the gauge invariant states is also true.

CHAPTER 5

DYNAMICAL SYSTEMS

In this chapter we introduce the notion of KMS states [24,36]. Utilizing the geometric approach to the Tomita-Takesaki theory of Modular Hilbert algebras due to Rieffel and Van Daele [51], we shall relate the KMS condition to an underlying modular structure. We shall then show that the KMS condition is equivalent to Sewell's inequality [52] (modulo some technicalities) and implies Bogolubov's inequality [53].

The theory of Rieffel and Van Daele is sufficiently general that we have been able to proceed by considering general topological algebras.

More precisely, in this section A will denote a locally convex Hausdorff unital topological $*$ -algebra, δ a distinguished $*$ -derivation on A , assumed continuous for simplicity. The pair (A, δ) will be referred to as a dynamical system.

5.1 The Rieffel-Van Daele Theory

Let us start by describing the Rieffel-Van Daele results in a form convenient for us [51].

Proposition 5.1 Let \mathcal{H} be a complex separable Hilbert space and \mathcal{K} a closed real subspace such that $\mathcal{K} \cap i\mathcal{K} = \{0\}$ and $\mathcal{K} + i\mathcal{K}$ is dense in, but not equal to \mathcal{H} . Then there exists a strongly continuous one-parameter unitary group $\{\Delta^{it} : t \in \mathbb{R}\}$ on \mathcal{H} such that:

- (i) $\Delta^{it}\mathcal{K} = \mathcal{K}$ for all $t \in \mathbb{R}$;
- (ii) to every pair $\phi, \psi \in \mathcal{K}$ there is a function $F : \mathbb{C} \rightarrow \mathbb{C}$ which is analytic in the strip $S(-1,0)$, bounded and continuous on its closure and taking boundary values

$$F(t) = \langle \phi, \Delta^{it}\psi \rangle$$

$$F(t-i) = \langle \Delta^{it} \psi, \phi \rangle \quad (5.1)$$

The unitary group $\{\Delta^{it} : t \in \mathbb{R}\}$ is the unique such group satisfying the above two conditions.

Let $P : \mathcal{H} \rightarrow \mathcal{K}$, $Q : \mathcal{H} \rightarrow i\mathcal{K}$ be the indicated real orthogonal projections, and set $R = P+Q$. Then the modular group is related to the subspace projections by

$$\Delta^{it} = (2-R)^{it} R^{-it} \quad (t \in \mathbb{R}) \quad (5.2)$$

We have introduced the notation

$$S(a,b) = \{z \in \mathbb{C} : a < \text{Im} z < b\} \quad (5.3)$$

By $A_n(a,b)$ we shall mean analyticity in $S(a,b)$; boundedness and continuity in its closure; and boundary values explicitly stated. Although this makes for extra notation, these analyticity conditions occur frequently enough to make the abbreviation worthwhile.

It is shown in [51] that if M is a von Neumann algebra with a cyclic and separating vector ω , acting on \mathcal{H} and M_s denotes the collection of self-adjoint elements of M , then the closure of $[M_s \omega]$ ($= \mathcal{K}$) is a closed real subspace of \mathcal{H} such that $\mathcal{K} \cap i\mathcal{K} = \{0\}$ and $\mathcal{K} + i\mathcal{K}$ is dense in \mathcal{H} . In Appendix D we describe another procedure to construct a closed real subspace with the required properties, independently of von Neumann algebra considerations.

5.2 The KMS Condition

The following lemma concerning the implementability of derivations is standard, c.f. Proposition 3.12.

Lemma 5.2 A state ϕ on the dynamical system (\mathcal{A}, δ) is said to be stationary if $\phi \circ \delta = \phi$. In the GNS representation of a stationary state ϕ there exists a densely defined symmetric operator H_ϕ for which

$$H_\phi \Omega_\phi = 0 \quad (5.4)$$

$$\pi_\phi(\delta(f))\Phi = [H_\phi, \pi_\phi(f)]\Phi$$

for all $f \in \mathcal{A}$, $\Phi \in \mathcal{D}_\phi$. ■

We shall now define KMS states. It will be noted that as δ is not required to be an infinitesimal generator causes us to introduce the suppositions that H_ϕ is essentially self-adjoint and that the real subspace \mathcal{K}_ϕ is temporally stable.

Definition 5.3 A state ϕ on the dynamical system (\mathcal{A}, δ) is said to be β -KMS if:

- (i) ϕ is stationary;
- (ii) the implementing operator H_ϕ is essentially self-adjoint;
- (iii) Let \mathcal{K}_ϕ be the closed real subspace

$$\mathcal{K}_\phi = \text{cl}\{[f]_\phi : f \in \mathcal{A}_h\} \quad (5.5)$$

and let $\{\tau_t^\phi : t \in \mathbb{R}\}$ be the unitary group generated by H_ϕ^{**} . Then $\tau_t^\phi \mathcal{K}_\phi = \mathcal{K}_\phi$ for all $t \in \mathbb{R}$.

- (iv) For every pair $f, g \in \mathcal{A}$ there is a function F_{fg} satisfying $An(0, \beta)$ with boundary values

$$\begin{aligned} F_{fg}(t) &= \langle [f]^*]_\phi, \tau_t^\phi [g]_\phi \rangle \\ F_{fg}(t+i\beta) &= \langle \tau_t^\phi [g]^*]_\phi, [f]_\phi \rangle \end{aligned} \quad (5.6)$$

The following technical lemma is necessary in order to apply the Rieffel-Van Daele theory. For a proof see [51], Theorem 3.9.

Lemma 5.4 Let $\{W_n\}_{n \in \mathbb{N}}$ be a sequence of functions satisfying $An(0, \beta)$. Suppose that in the sup-norm $W_n(t) \longrightarrow G(t)$ and $W_n(t+i\beta) \longrightarrow H(t)$. Then there exists a function W satisfying $An(0, \beta)$ with boundary values $W(t) = G(t)$, $W(t+i\beta) = H(t)$. ■

Corollary 5.5 Let ϕ be a β -KMS state on (\mathcal{A}, δ) . To every pair $\Phi, \Psi \in \mathcal{K}_\phi$ there exists a function F satisfying $An(0, \beta)$ with boundary values

$$F(t) = \langle \Phi, \tau_t^\phi \Psi \rangle \quad (5.7)$$

$$F(t+i\beta) = \langle \tau_t^\phi \Psi, \Phi \rangle$$

Lemma 5.6 For a β -KMS state ϕ on (\mathcal{A}, δ) , the subspace \mathcal{K}_ϕ satisfies $\mathcal{K}_\phi \cap i\mathcal{K}_\phi = \{0\}$.

Proof If $\Psi \in \mathcal{K}_\phi \cap i\mathcal{K}_\phi$, then $\Psi, i\Psi \in \mathcal{K}_\phi$. By Corollary 5.5 for all $\Phi \in \mathcal{K}_\phi$ there exist functions F, G satisfying $An(0, \beta)$ with boundary values

$$F(t) = \langle \Phi, \tau_t^\phi \Psi \rangle, \quad F(t+i\beta) = \langle \tau_t^\phi \Psi, \Phi \rangle$$

$$G(t) = \langle \Phi, \tau_t^\phi i\Psi \rangle, \quad G(t+i\beta) = \langle \tau_t^\phi i\Psi, \Phi \rangle$$

The function $H = F + iG$ satisfies $An(0, \beta)$ with boundary values

$$H(t) = 0 \quad H(t+i\beta) = 2F(t+i\beta)$$

By [54], Theorem 12.8, $H=0$. Then $F(t+i\beta) = 0 = \langle \tau_t^\phi \Psi, \Phi \rangle$ and setting $t=0$, $\Phi = \Psi$, the desired conclusion follows. ■

Proposition 5.7 For a β -KMS state ϕ on (\mathcal{A}, δ) , Ω_ϕ is separating for $\pi_\phi(\mathcal{A})$.

Proof First we prove that if $\phi(f^*f) = 0$, then $\phi(ff^*) = 0$. Assuming $\phi(f^*f) = 0$, $f = f_1 + if_2$, $f_1, f_2 \in \mathcal{A}_h$, we get $[f_1]_\phi + i[f_2]_\phi = 0$. By Lemma 5.6, $[f_1]_\phi = [f_2]_\phi = 0$, so $\phi(ff^*) = 0$.

If $\pi_\phi(f)\Omega_\phi = 0$, then $\pi_\phi(gf)\Omega_\phi = \pi_\phi(g)\pi_\phi(f)\Omega_\phi = 0$, or equivalently, $\phi(f^*g^*gf) = 0$, for all $g \in \mathcal{A}$. Therefore $\phi(gff^*g^*) = 0$, or $\pi_\phi(f^*)\pi_\phi(g^*)\Omega_\phi = 0$, for all $g \in \mathcal{A}$. Since Ω_ϕ is cyclic and $\pi_\phi(f^*)$ closeable, we get that $\pi_\phi(f^*) = 0$. Taking adjoints gives $\pi_\phi(f^*)^* = 0$, which implies $\pi_\phi(f) = 0$, because $\pi_\phi(f) \subset \pi_\phi(f^*)^*$. ■

The modular structure arises from a conjugation on $\mathcal{K}_\phi + i\mathcal{K}_\phi$. We now prove that the required conjugation is closed.

Lemma 5.8 For a β -KMS state ϕ on (\mathcal{A}, δ) , let S_ϕ be the operator with domain $\mathcal{K}_\phi + i\mathcal{K}_\phi$ and

$$S_\phi(\phi + i\psi) = \phi - i\psi \quad (\phi, \psi \in \mathcal{K}_\phi) \quad (5.8)$$

Then S_ϕ is densely defined and closed.

Proof Assume $\phi_n + i\psi_n \rightarrow 0$, and $\phi_n - i\psi_n$ convergent, where $\phi_n, \psi_n \in \mathcal{K}_\phi$. Since \mathcal{K}_ϕ is closed, and ϕ_n, ψ_n convergent, there exist $\phi, \psi \in \mathcal{K}_\phi$ such that $\phi_n \rightarrow \phi$ and $\psi_n \rightarrow \psi$. Now $\phi_n + i\psi_n \rightarrow 0$ gives $\phi, \psi \in \mathcal{K}_\phi \cap i\mathcal{K}_\phi$, which implies $\phi = \psi = 0$. Consequently $\phi_n - i\psi_n \rightarrow 0$, so S_ϕ is closed. ■

Definition 5.9 With the notation as above, the modular operator associated with the β -KMS state ϕ is given by

$$\Delta_\phi = S_\phi^* S_\phi \quad (5.9)$$

From this, Δ_ϕ is strictly positive and self-adjoint. The polar decomposition of S_ϕ ,

$$S_\phi = J_\phi \Delta_\phi^{1/2} \quad (5.10)$$

then serves to define the modular conjugation J_ϕ , an anti-unitary involution.

Proposition 5.10 Let ϕ be a β -KMS state on (\mathcal{A}, δ) , Δ_ϕ the modular operator, and $\{\tau_t^\phi : t \in \mathbb{R}\}$ the unitary group generated by the implementing Hamiltonian H_ϕ^{**} . Then

$$\tau_{-it}^\phi = \Delta_\phi^{it} \quad (5.11a)$$

or

$$H_\phi^{**} = -\beta^{-1} \ln \Delta_\phi \quad (5.11b)$$

Proof Recall that τ_{-t}^ϕ leaves \mathcal{K}_ϕ invariant for all $t \in \mathbb{R}$. For every pair $\Phi, \Psi \in \mathcal{K}_\phi$ there is a function F satisfying $A_n(-1, 0)$ with boundary values

$$F(t) = \langle \Phi, \tau_{-t}^\phi \Psi \rangle$$

$$F(t-i) = \langle \tau_{-t}^\phi \Psi, \Phi \rangle$$

By the uniqueness result mentioned in Proposition 5.1, equation (5.11a), and hence (5.11b), follows. ■

5.3 Correlation Inequalities

We now turn our attention to the inequalities of Sewell [52] and Bogolubov [53].

Lemma 5.11 Let H be a self-adjoint operator on a separable Hilbert space, and $v \in \text{dom}(H) \cap \text{dom}(\rho_\beta)$, where $\rho_t = \exp(-tH/2)$ for $t \in \mathbb{R}$. Then

$$\beta \langle v, Hv \rangle \geq \|v\|^2 \ln(\|v\|^2 \|\rho_\beta v\|^{-2}) \quad (5.12)$$

Proof Recall that Jensen's inequality ([54]), Theorem 3.3) states if μ is a probability measure, $f \in L^1(\mu)$ real and with range in (a,b) , and ϕ convex on (a,b) , then

$$\phi\left(\int f d\mu\right) \leq \int (\phi \circ f) d\mu \quad (5.13)$$

Now if $H = \int_{-\infty}^{\infty} \lambda dE(\lambda)$ is the spectral resolution of H , then

$$\langle v, \phi(H)v \rangle = \int_{-\infty}^{\infty} \phi(\lambda) d\langle v, E(\lambda)v \rangle$$

with $f=1$ and $d\mu = ||v||^{-2} d\langle v, E(\lambda)v \rangle$, Jensen's inequality gives

$$\phi(||v||^{-2} \langle v, Hv \rangle) \leq ||v||^{-2} \langle v, \phi(H)v \rangle$$

Taking $\phi(t) = \exp(-\beta t/2)$, squaring, using the Cauchy-Schwarz inequality, taking logarithms, and multiplying by -1 gives the result. ■

Proposition 5.12 A β -KMS state ϕ on (\mathcal{A}, δ) obeys Sewell's inequality

$$-\beta \phi(\delta(f^*)f) \geq \phi(f^*f) \ln[\phi(f^*f)\phi(ff^*)^{-1}] \quad (5.14)$$

Conversely if a state ϕ satisfies Sewell's inequality, then it is stationary. If in addition the self-adjointness and stability conditions (ii), (iii) of Definition 5.3 are satisfied, ϕ is β -KMS.

Proof Take $H = H_{\phi}^{**}$ in the previous lemma, and $v = [f]_{\phi}$. Then

$||v||^2 = \phi(f^*f)$; $\langle v, Hv \rangle = -\phi(\delta(f^*)f)$; and

$$||\rho_{\beta} v||^2 = ||\Delta_{\phi}^{1/2} v||^2 = ||J_{\phi} \Delta_{\phi}^{1/2} v||^2 = ||[f^*]_{\phi}||^2 = \phi(ff^*)$$

The inequality (5.12) gives (5.14) with these values substituted.

For the converse we start by showing stationarity. From Sewell's inequality follows

$$\operatorname{Im} \phi(\delta(f^*)f) = 0$$

For $f = g^*-1$, we get $\operatorname{Im} \phi(\delta(g)) = 0$. Replacing g by ig gives $\operatorname{Re} \phi(\delta(g)) = 0$, so $\phi(\delta(g)) = 0$. Now assume (ii), (iii) of Definition 5.3 in addition.

For $k \in C_c(\mathbb{R})$ consider

$$\nu(k) = \int k(-\lambda) d\langle [f^*]_\phi, E(\lambda)[f^*]_\phi \rangle$$

$$\mu(k) = \int k(\lambda) d\langle [f]_\phi, E(\lambda)[f]_\phi \rangle$$

Sewell ([52], Theorem 3) has shown that $\nu < \mu$ and the Radon-Nikodym derivative is $d\nu/d\mu(\lambda) = \exp(-\beta\lambda)$. Here $\{E(\lambda) : \lambda \in \mathbb{R}\}$ is the spectral family for H_ϕ^{**} . The remainder of Sewell's proof that ϕ is β -KMS now goes through : ibid Lemma 12. ■

To discuss Bogolubov's inequality we need the following lemma on operator bounds.

Lemma 5.13 Let Δ be a positive self-adjoint operator and $0 < \alpha < 1$. Then for every $\Psi \in \operatorname{dom}(\Delta)$,

$$||\Delta^\alpha \Psi||^2 \leq \alpha ||\Delta \Psi||^2 + (1-\alpha) ||\Psi||^2 \quad (5.15)$$

If $\Phi, \Psi \in \operatorname{dom}(\Delta^\alpha) \cap \operatorname{dom}(\Delta^\alpha \ln \Delta)$, we also have

$$\frac{d}{d\alpha} \langle \Delta^\alpha \Phi, \Delta^\alpha \Psi \rangle = \langle \Delta^\alpha \ln \Delta \Phi, \Delta^\alpha \Psi \rangle + \langle \Delta^\alpha \Phi, \Delta^\alpha \ln \Delta \Psi \rangle \quad (5.16)$$

Proof From Holder's inequality for the spectral resolution,

$$||\Delta^\alpha \Psi||^2 \leq ||\Delta \Psi||^{2\alpha} ||\Psi||^{2-2\alpha}$$

and by Jensen's inequality ([54], p.64) .

$$||\Delta \Psi||^{2\alpha} ||\Psi||^{2-2\alpha} \leq \alpha ||\Delta \Psi||^2 + (1-\alpha) ||\Psi||^2,$$

giving the first result.

For the second result we consider the derivative as a limit. We write

$$\begin{aligned} h^{-1} \langle \Delta^{\alpha+h} \psi, \Delta^{\alpha+h} \phi \rangle &= h^{-1} \langle \Delta^{\alpha} \psi, \Delta^{\alpha} \phi \rangle \\ &= h^{-1} \langle (\Delta^{\alpha+h} - \Delta^{\alpha}) \psi, \Delta^{\alpha+h} \phi \rangle + h^{-1} \langle \Delta^{\alpha} \psi, (\Delta^{\alpha+h} - \Delta^{\alpha}) \phi \rangle \end{aligned}$$

and since the scalar product is jointly continuous we can pass to the limit $h \rightarrow 0$. ■

Proposition 5.14 A β -KMS state ϕ on (\mathcal{A}, δ) obeys Bogolubov's inequality

$$|\phi([f, g])|^2 \leq (\beta/2) \phi(f^* f + f f^*) \phi([g^*, \delta(g)]).$$

Proof We follow the proof given in Powers [53], Theorem 1.

Writing $\rho_t = \exp(-tH_{\phi}^{**}/2)$, let \mathcal{D}_1 be \mathcal{D}_{ϕ} equipped with the inner product

$$\langle [f]_{\phi}, [g]_{\phi} \rangle_1 = \int_0^{\beta} \langle \rho_t [f]_{\phi}, \rho_t [g]_{\phi} \rangle dt;$$

note that by equation (5.15) $[f]_{\phi} \in \text{dom}(\rho_t)$ for $0 \leq t \leq \beta$ since

$$\| \rho_{\beta} [f]_{\phi} \| = \| [f^*]_{\phi} \|$$

From the previous lemma, equation (5.15) we get

$$\| [f]_{\phi} \|_1^2 \leq (\beta/2) \phi(f^* f + f f^*)$$

and from equation (5.16)

$$\langle [f]_{\phi}, [\delta(g)]_{\phi} \rangle_1 = \phi([f^*, g])$$

$$\text{Then } |\phi([f, g])|^2 \leq \| [f^*]_{\phi} \|_1^2 \| [\delta(g)]_{\phi} \|_1^2$$

$$\leq (\beta/2) \phi(f^* f + f f^*) \phi([\delta(g)^*, g])$$

The result follows since ϕ is a hermitian functional. ■

In our work so far, we have not considered the Green's functions. It seems to us that their existence must be considered separately for different models, e.g., the conditions to ensure existence for the ideal Bose gas will be rather different from those for the continuum Heisenberg ferromagnet. They should lead to the same conclusion, however, that for a state ϕ in which δ is implemented, and generates a $\tau_t^\phi = \exp(itH_\phi^{**})$ that

$$\tau_t^\phi(D_\phi^{**}) = D_\phi^{**}$$

For then we can define

$$G^\phi(f_1, t_1, \dots, f_n, t_n) = \langle \Omega_\phi, \tau_{t_1}^\phi \pi_\phi^{**}(f_1) \tau_{t_2 - t_1}^\phi \dots \tau_{t_n - t_{n-1}}^\phi \pi_\phi^{**}(f_n) \tau_{t_n}^\phi \Omega_\phi \rangle$$

We leave this question open.

APPENDIX A
LOCALLY CONVEX SPACES

For completeness in this Appendix we shall summarize the definitions, and the results from the theory of locally convex spaces that have been employed in the main body of the thesis. We will only supply proofs for those results that we have not been able to find in the literature. Material for which we have not given any references may be found in the standard textbooks: [12], [13], [15]. All vector spaces are assumed to be complex.

Definition A.1 A seminorm on a vector space E is a map $p : E \longrightarrow [0, \infty)$ obeying

- (i) $p(f+g) \leq p(f) + p(g)$
- (ii) $p(\lambda f) = |\lambda|p(f)$ for $\lambda \in \mathbb{C}$

A family of seminorms $\{p_i\}_{i \in I}$ is said to separate points if

- (iii) $p_i(f) = 0$ for all $i \in I$ implies $f=0$

Definition A.2 A set in a vector space E is convex if $f, g \in A$, $0 \leq \lambda \leq 1$, implies $\lambda f + (1-\lambda)g \in A$. A is balanced (or circled) if $f \in A$ and $|\lambda| \leq 1$ implies $\lambda f \in A$. Finally, A is absorbing if for every $f \in E$, $\mu f \in A$ for some $\mu > 0$.

Definition A.3 The gauge of an absorbing set A in a vector space E is the function $f \longmapsto p_A(f) = \inf\{\lambda > 0 : f \in \lambda A\}$.

Proposition A.4 The gauge of an absorbing balanced convex set A is a seminorm. Conversely, a seminorm p is the gauge of any absorbing balanced convex set A such that

$$\{f : p(f) < 1\} \subset A \subset \{f : p(f) \leq 1\}$$

Definition A.5 A locally convex space (lcs) is a vector space E with a family of seminorms $\{p_i\}_{i \in I}$ separating points and equipped with the coarsest topology \mathcal{T} for which all the sets

$$f + \mathcal{U}(i_1, \dots, i_n; E), f \in E, n \in \mathbb{N}, \mathcal{U}(i_1, \dots, i_n; \varepsilon) = \{f : p_{i_1}(f) < \varepsilon, \dots, p_{i_n}(f) < \varepsilon\}$$

are open.

Definition A.6 A family of continuous seminorms \mathcal{P} on a lcs E is a generating family of seminorms if to any continuous seminorm q there is a seminorm $p \in \mathcal{P}$ and $c > 0$ such that for all $f \in E$

$$q(f) \leq cp(f)$$

Definition A.7 A closed absorbing convex balanced set A in a lcs E is a barrel.

Proposition A.8 The gauge of every barrel is lower semicontinuous.

Definition A.9 A lcs E is barrelled if every barrel is a neighbourhood of zero.

Definition A.10 A set B in a lcs E is bounded if $\sup_{f \in B} |p(f)| < +\infty$ for every continuous seminorm p .

Definition A.11 A Montel space E is a barrelled lcs in which every closed bounded set is compact.

Definition A.12 A lcs is bornological if every seminorm that is bounded on every bounded set is continuous.

Definition A.13 A dual pair $\langle E, F \rangle$ is a vector space E and a space F of linear functionals on E which separates points.

Definition A.14 Let $\langle E, F \rangle$ be a dual pair. The F -weak topology on E , $\sigma(E, F)$ is the locally convex topology generated by the family of seminorms $\{p_\phi : p_\phi(f) = |\phi(f)|, \phi \in F\}$.

Definition A.15 Let $\langle E, F \rangle$ be a dual pair. The Mackey topology on E , $\tau(E, F)$, is the locally convex topology generated by the family of seminorms $\{p_C : p_C(f) = \sup_{\phi \in C} |\phi(f)|, C \subset F \text{ and } \sigma(F, E)\text{-compact}\}$

Definition A.16 The topological dual E' of a lcs E is the set of all continuous linear functionals on E .

Proposition A.17 If E is a lcs $\langle E, E' \rangle$ is a dual pair.

Definition A.18 Let $\langle E, F \rangle$ be a dual pair. A locally convex topology \mathcal{T} on E is a dual - $\langle E, F \rangle$ topology if the topological dual of $E[\mathcal{T}]$ is F .

Proposition A.19 (Mackey-Arens) let $\langle E, F \rangle$ be a dual pair. A locally convex topology \mathcal{T} on E is a dual - $\langle E, F \rangle$ topology iff

$$\sigma(E, F) \subset \mathcal{T} \subset \tau(E, F)$$

Proposition A.20 Let $\langle E, F \rangle$ be a dual pair. All dual - $\langle E, F \rangle$ topologies have the same bounded sets and the same closed convex sets.

Proposition A.21 If $E[\mathcal{T}]$ is a barrelled or bornological lcs then

$$\mathcal{T} = \tau(E, E').$$

Definition A.22 Let $\langle E, F \rangle$ be a dual pair and let $A \subset E$. The polar of A , A° , is $\{\phi \in F : |\phi(f)| \leq 1 \text{ for all } f \in A\}$.

Definition A.23 Let E be a lcs. A set C in E' is equicontinuous if $C \subset A^\circ$ for some neighbourhood A of zero.

Definition A.24 ([14]) A lcs E is a nuclear (resp. Schwartz) space if for every continuous seminorm p there is a summable (resp. zero convergent) sequence of positive numbers $\{\lambda_n\}_{n \in \mathbb{N}}$ and an equicontinuous sequence $\{\phi_n\}_{n \in \mathbb{N}}$ in E' such that

$$p(f)^2 \leq \sum_{n \in \mathbb{N}} \lambda_n |\phi_n(f)|^2 \quad (\text{resp. } p(f) \leq \sup_{n \in \mathbb{N}} \lambda_n |\phi_n(f)|)$$

for all $f \in E$.

Definition A.25 Let E be a lcs and E' its dual. The strong topology $\beta(E', E)$ on E' is the locally convex topology generated by the family of seminorms $\{p_B : p_B(\phi) = \sup_{f \in B} |\phi(f)|, B \subset E \text{ and bounded}\}$

Definition A.26 A lcs E is dual nuclear or conuclear if $E'[\beta(E', E)]$ is nuclear.

Definition A.27 A lcs $E[\mathcal{T}]$ is reflexive if the $\beta(E', E)$ dual of E' is E and $\beta(E, E') = \mathcal{T}$.

Definition A.28 A directed system is a set Λ together with an ordering $<$ which satisfies:

- (i) if $\alpha, \beta \in \Lambda$, then there exists $\gamma \in \Lambda$ so that $\alpha < \gamma$ and $\beta < \gamma$.
- (ii) $<$ is a partial ordering.

Definition A.29 A net in a topological space E is a mapping from a directed system Λ to E ; we denote it by $\{f_\alpha\}_{\alpha \in \Lambda}$.

Definition A.30 A net $\{f_\alpha\}_{\alpha \in \Lambda}$ in a topological space E converges to a point $f \in E$ (written $f_\alpha \longrightarrow f$) if for every neighbourhood U of f , there is a $\beta \in \Lambda$ so that $f_\alpha \in U$ if $\beta < \alpha$.

Definition A.31 A net $\{f_\alpha\}_{\alpha \in \Lambda}$ in a lcs E is a Cauchy net if for every $\epsilon > 0$, and for every continuous seminorm p there is a $\gamma \in \Lambda$ such that $p(f_\alpha - f_\beta) < \epsilon$ if $\gamma < \alpha, \beta$. E is complete if every Cauchy net converges.

Definition A.32 A lcs is quasi-complete if every bounded, closed subset of E is complete.

Proposition A.33 (Dieudonné-Schwartz) Let E be a vector space and let $\{E_n\}_{n \in \mathbb{N}}$ be a sequence of linear subspaces of E such that $E_n \subset E_{n+1}$ for all $n \in \mathbb{N}$ and $E = \bigcup_{n \in \mathbb{N}} E_n$. Suppose that each E_n is equipped with a locally convex topology \mathcal{T}_n and for each n the topology induced by \mathcal{T}_{n+1} on E_n is \mathcal{T}_n . Let \mathcal{T} be the finest locally convex topology on E for which all canonical injections $i_n : E_n \longrightarrow E$ are continuous. Then \mathcal{T} induces on each E_n the topology \mathcal{T}_n .

Definition A.34 If the hypotheses of Proposition A.33 are satisfied, then we say that E is the strict inductive limit of the sequence $\{E_n\}_{n \in \mathbb{N}}$ and that $\{E_n\}_{n \in \mathbb{N}}$ is a defining sequence of E and write $E = \text{strict-}\varinjlim E_n$.

Proposition A.35 (Dieudonné-Schwartz) Let E be the strict inductive limit of the sequence $\{E_n\}_{n \in \mathbb{N}}$ and let E_n be closed in E_{n+1} for all $n \in \mathbb{N}$. Then a set B in E is bounded iff B is contained in some E_n and is bounded there.

Definition A.36 A Fréchet space or F -space is a complete metrizable lcs.

Definition A.37 An LF-space is a strict inductive limit of F-spaces. LF-spaces that are strict inductive limits of Banach spaces are known as LB-spaces.

Proposition A.38 Let E be an LF-space. A linear map of E into a lcs is continuous iff it is sequentially continuous.

Definition A.39 Let E_1, E_2, G be lcs. A bilinear map $\omega : E_1 \times E_2 \longrightarrow G$ is separately continuous if the maps $f_1 \longmapsto \omega(f_1, g)$ and $f_2 \longmapsto \omega(h, f_2)$ are continuous for all $g \in E_2, h \in E_1$.

Definition A.40 Let E_1, E_2, G be lcs. A bilinear map $\omega : E_1 \times E_2 \longrightarrow G$ is hypocontinuous if its restrictions to $B_1 \times E_2$ and $E_1 \times B_2$ are continuous, for all B_1, B_2 bounded.

Proposition A.41 Let E_1, E_2, G be lcs with E_1, E_2 barrelled. Then every separately continuous bilinear map $\omega : E_1 \times E_2 \longrightarrow G$ is hypocontinuous.

Definition A.42 If E_1 and E_2 are lcs the inductive (resp. projective) tensor product topology ι (resp. π) is the finest locally convex topology on $E_1 \otimes E_2$ for which the canonical bilinear map $\chi : E_1 \times E_2 \longrightarrow E_1 \otimes E_2, \chi(f_1, f_2) = f_1 \otimes f_2$, is separately continuous (resp. continuous). The tensor product equipped with the ι (resp. π)-topology is denoted by $E_1 \otimes_{\iota} E_2$ (resp. $E_1 \otimes_{\pi} E_2$).

Proposition A.43 The ι -topology is finer than the π -topology, but they coincide when both factors of the tensor product are Fréchet spaces, DF-barrelled, or LB-spaces. ([49], Chapter I, p.74; [55], p.316).

Definition A.44 The completion of the lcs $E \otimes F$ in the ι (resp. π)-topology is called the completed inductive (resp. projective) tensor product of the lcs E and F . It is denoted by $E \bar{\otimes} F$ (resp. $E \hat{\otimes} F$).

Proposition A.45 If E and F are F -spaces, $E \hat{\otimes} F$ is an F -space. If E and F are nuclear, $E \otimes_{\pi} F$ is nuclear.

Proposition A.46 Assume that E, F are the inductive limits of the lcs $\{E_i : i \in I\}$, $\{F_j : j \in J\}$, respectively. Then $E \bar{\otimes} F$ is the inductive limit of the lcs $\{E_i \bar{\otimes} F_j : i \in I, j \in J\}$. [56], Theorem A 2.2.5)

Proposition A.47 Let E_i, F_i ($i=1,2$) be lcs and let $A_i : E_i \longrightarrow F_i$ be continuous linear maps. Then $A_1 \otimes A_2 : E_1 \otimes_1 E_2 \longrightarrow F_1 \otimes_1 F_2$, is continuous. By extension we get a continuous linear map $A_1 \bar{\otimes} A_2 : E_1 \bar{\otimes} E_2 \longrightarrow F_1 \bar{\otimes} F_2$. ([49], Chapter I, p.75).

Proposition A.48 Let E and F be lcs. If $\{a_{\alpha}\}_{\alpha \in \Lambda}$ and $\{b_{\delta}\}_{\delta \in \Delta}$ are convergent nets to a in E and b in F , respectively, then $\{a_{\alpha} \otimes b_{\delta}\}_{\alpha \in \Lambda, \delta \in \Delta}$ and $\{a \otimes b_{\delta}\}_{\delta \in \Delta}$ are nets that converge to $a \otimes b$ in $E \otimes_1 F$.

Proposition A.49 Let E and F be nuclear LF-spaces. Then $E \bar{\otimes} F$ is a nuclear LF-space.

Proof By Proposition A.46 $E \bar{\otimes} F$ is the inductive limit of the Fréchet spaces $\{E_k \bar{\otimes} F_k = E_k \hat{\otimes} F_k : k \in \mathbb{N}\}$, where $\{E_k : k \in \mathbb{N}\}$ and $\{F_k : k \in \mathbb{N}\}$ are defining sequences for E and F respectively. Since $\{E_k\}$ and $\{F_k\}$ are nuclear, $E_{k+1} \hat{\otimes} F_{k+1}$ induces the original topology on $E_k \hat{\otimes} F_k$ ([57], p.119). Then $E \bar{\otimes} F = \text{strict-}\lim_{\rightarrow} E_k \hat{\otimes} F_k$ is LF and nuclear ([13], Proposition 50.1). ■

Proposition A.50 A locally convex direct sum E of a sequence $\{E_i\}$ of LF-spaces is LF.

Proof * Let $E = \bigoplus_{i=1}^{\infty} E_i$, and $E_i = \text{strict-lim} \rightarrow A_{ij}$. Define now

$A_k = \bigoplus_{j=1}^k A_{jk}$. The $\{A_k\}$ are Fréchet spaces such that $\bigcup_{k \in \mathbb{N}} A_k = E$,

$A_k \subset A_{k+1}$, and A_{k+1} induces the original topology on A_k ; $\text{strict-lim} \rightarrow$

A_k defines then a bornological topology on E that has the same bounded sets as the original bornological topology on E (c.f. Proposition A.35).

By [12], Proposition 8.3, p.62, both topologies are equal and consequently E is an LF-space. ■

Proposition A.51 Let \mathcal{P}_i ($i=1,2$) be generating families of semi-

norms for the lcs E_i . Then $\{p_1 \otimes_{\pi} p_2 : p_1 \in \mathcal{P}_1, p_2 \in \mathcal{P}_2\}$, where

$(p_1 \otimes_{\pi} p_2)(h) = \inf \left\{ \sum_{r=1}^n p_1(f_r) p_2(g_r) \right\}$ and the infimum is taken over all

possible representations of the element h in the form $h = \sum_{r=1}^n f_r \otimes g_r$,

is a generating family of seminorms for $E_1 \otimes_{\pi} E_2$.

Similarly $\{p_1 \otimes_{\epsilon} p_2 : p_1 \in \mathcal{P}_1, p_2 \in \mathcal{P}_2\}$ where $(p_1 \otimes_{\epsilon} p_2) \left(\sum_{r=1}^n f_r \otimes g_r \right)$

$= \sup \left\{ \left| \sum_{r=1}^n \phi_1(f_r) \phi_2(g_r) \right| : \phi_1 \in B_{p_1}^{\circ}, \phi_2 \in B_{p_2}^{\circ} \right\}$ and $B_p = \{f : p(f) < 1\}$, is a

generating family of seminorms for a locally convex topology on $E_1 \otimes E_2$,

the injective tensor product topology ϵ . $E_1 \otimes E_2$ equipped with this

topology is denoted by $E_1 \otimes_{\epsilon} E_2$. For all $p_i \in \mathcal{P}_i$ ($i=1,2$) we have

$(p_1 \otimes_{\epsilon} p_2)(h) \leq (p_1 \otimes_{\pi} p_2)(h)$ for all $h \in E_1 \otimes E_2$.

Corollary A.52 The π -topology is finer than the ϵ -topology.

Proposition A.53 A lcs E is nuclear iff for every lcs F

$$E \otimes_{\pi} F = E \otimes_{\epsilon} F$$

APPENDIX B

ORDERED VECTOR SPACES

In this Appendix we briefly summarize the part of the theory of ordered vector spaces that has been used in Section 2. In what follows all vector spaces should be taken as real. We will only give references for results that do not appear in the texts [12] and [20].

Definition B.1 A non-empty convex subset C of a vector space E is a cone if $\lambda C \subset C$ for all $\lambda \geq 0$. A vector space E with a cone C will be called an ordered vector space (ovs) and denoted by (E, C) .

Proposition B.2 A cone C in E determines a transitive and reflexive relation \leq by,

$$f \leq g \text{ if } g - f \in C$$

This relation is compatible with the vector structure, i.e.,

- (i) if $f \geq 0$ and $g \geq 0$ then $f + g \geq 0$
- (ii) if $f \geq 0$ then $\lambda f \geq 0$ for all $\lambda \geq 0$

Conversely if \leq is a relation in E which is transitive, reflexive, and compatible with the vector structure and if we define

$$C = \{f \in E : f \geq 0\}$$

then C is a cone, and \leq is exactly the vector ordering of E induced by C .

Definition B.3 A cone C in E is proper if $C \cap -C = \{0\}$.

Proposition B.4 The vector ordering \leq of E , induced by a cone C , is antisymmetric iff C is proper.

Proposition B.5 The intersection of a family of cones in E is a cone.

Definition B.6 The smallest cone containing a subset A of a vector space E is $\text{pos } A$.

Definition B.7 A cone C is generating if $E = C - C$.

Definition B.8 An order-interval in an ovs E is a set of the form

$$[f, g] = \{h \in E : f \leq h \leq g\}$$

Definition B.9 A set A in an ovs E is order bounded if $A \subset [f, g]$, for some $f, g \in E$. $e \in E$ is an order unit if $[-e, e]$ is absorbing.

Definition B.10 Let $C \neq \{0\}$ be a cone in E . A non-empty convex subset B of C is a base for C if each non-zero element $f \in C$ has a unique representation of the form $f = \lambda g$, where $\lambda > 0$ and $g \in B$.

Definition B.11 A ray ρ of a cone $C \neq \{0\}$ is a set of the form

$$\{\lambda f : \lambda \geq 0, f \in C \text{ and } f \neq 0\}$$

Definition B.12 A ray ρ of a cone $C \neq \{0\}$ is extremal if $f \in \rho$ and $g \leq f$ implies $g \in \rho$.

Definition B.13 A cone C in a lcs E is normal if there is a generating family of seminorms \mathcal{P} on E such that

$$q(f) \leq q(f+g) \quad \text{for all } f, g \in C, q \in \mathcal{P}$$

Proposition B.14 Every normal cone is proper.

Proposition B.15 If C is a cone in a lcs E , then

$$C' = \{\phi \in E' : \phi(f) \geq 0 \text{ for all } f \in C\}$$

is a cone in E' . C' is called the dual cone of C .

Proposition B.16 If C is a normal cone in a lcs E , then C' is generating for E' .

Definition B.17 A cone C in a lcs E is a strict b-cone if every bounded set B is contained in a set of the form $B_1 \cap C - B_1 \cap C$, where B_1 is bounded.

Proposition B.18 ([22]) Let C be a normal cone in a nuclear space E . Then for every continuous seminorm p there is a summable sequence of positive numbers $\{\lambda_n\}_{n \in \mathbb{N}}$ and an equicontinuous sequence $\{\phi_n\}_{n \in \mathbb{N}}$ in C' such that

$$p(f) \leq \sum_{n \in \mathbb{N}} \lambda_n |\phi_n(f)|$$

for all $f \in E$.

Proposition B.19 ([58]) Let $C \neq \{0\}$ be a closed proper cone in a quasi-complete dual nuclear space E such that $[0, f]$ is compact for all $f \in C$. Then C is the closed convex hull of its extreme rays.

Proposition B.20 ([59]); [60] Corollary p.97) Let E be a lcs, $\{f_j : j \in J\}$ a family of vectors in E and $\{c_j : j \in J\}$ an accompanying family of real numbers. Then there is $\phi \in E'$ such that

$$\phi(f_j) \geq c_j \text{ for all } j \in J$$

iff $(0, 1) \notin \text{cl-pos}\{(f_j, c_j) : j \in J\} \subset E \times \mathbb{R}$ (c.f. Definition B.6; cl = closure).

APPENDIX C

SEMIGROUPS IN LOCALLY CONVEX SPACES

We give in this Appendix a short review of the theory of semigroups in ℓcs : [61], [62]. We will only supply proofs of Propositions not appearing in these papers.

Let E be a ℓcs and $L(E)$ the set of continuous linear operators in E . A family M in $L(E)$ is said to be equicontinuous, if for any continuous seminorm p on E , there is a continuous seminorm q on E such that $p(Tf) \leq q(f)$ for all $T \in M$, $f \in E$.

Definition C.1 A one-parameter family $\{T_t : t \geq 0\}$ in $L(E)$ is called a semigroup, if it satisfies the following conditions:

- (1) $T_t T_s = T_{t+s}$ for all $t, s \geq 0$
- (2) $T_0 = I$ (the identity operator)
- (3) $\lim_{t \rightarrow s} T_t f = T_s f$ for all $s \geq 0$, $f \in E$

A semigroup $\{T_t : t \geq 0\}$ is said to be locally equicontinuous if for every $s \in (0, \infty)$, the subfamily $\{T_t : 0 \leq t \leq s\}$ is equicontinuous in $L(E)$.

We note that by the continuity of $T_t f$, the Riemann integral $\int_0^s T_t f dt$ ($0 < s < \infty$) exists in the sequential completion of E .

Proposition C.2 If E is barrelled, then every semigroup $\{T_t : t \geq 0\}$ in E is locally equicontinuous.

Definition C.3 The infinitesimal generator A of a semigroup

$\{T_t : t \geq 0\}$ is defined by

$$Af = \lim_{h \rightarrow 0} h^{-1} (T_h - I)f$$

whenever the limit exists in E .

Proposition C.4 Let $\{T_t : t \geq 0\}$ be a semigroup in a lcs E .

- (1) if $f \in D(A)$, then $T_t f \in D(A)$ for all $t \geq 0$ and $T_t f$ is continuously differentiable in t relative to the topology of E , and

$$\frac{d}{dt} T_t f = A T_t f = T_t A f \quad \text{for all } t \geq 0$$

- (2) An element f in E belongs to $D(A)$ and $Af = g$ iff

$$T_t f - f = \int_0^t T_s g \, ds \quad \text{for all } t \geq 0$$

Note that in this Proposition the sequential completeness of E is not assumed.

Corollary C.5 Let E be a sequentially complete lcs and let $\{T_t : t \geq 0\}$ be a semigroup in E . Then for every $f \in E$, $\int_a^b T_s f \, ds$ ($0 \leq a, b \leq \infty$) belongs to $D(A)$ and we have

$$A \int_a^b T_s f \, ds = T_b f - T_a f$$

Proposition C.6 Let E be a sequentially complete lcs. Then for every semigroup $\{T_t : t \geq 0\}$ in E , the domain $D(A)$ of its infinitesimal generator A is dense in E .

Proposition C.7 For every locally equicontinuous semigroup $\{T_t : t \geq 0\}$ in a lcs E , its infinitesimal generator A is closed.

Proposition C.8 Let E be a lcs and $\{T_t : t \geq 0\}$ a semigroup in E with continuous infinitesimal generator A , then if

$$R_n(t) = [(n-1)!]^{-1} t^n \int_0^1 (1-s)^{n-1} (A^n T_{st} f - A^n f) ds, \text{ we have}$$

$$T_t f = \sum_{k=0}^n (k!)^{-1} t^k A^k f + R_n(t) \text{ and } \lim_{t \rightarrow 0} t^{-n} R_n(t) = 0.$$

Proof This is just a particular case of Taylor's Theorem : [63]

Theorem A.4.1 and Corollary A.4.3. ■

The next Proposition is due to Prof. S. Ōuchi (private communication).

Proposition C.9 Let $\{E_n\}_{n \in \mathbb{N}}$ be complete, barrelled and $E = \bigoplus_{n=1}^{\infty} E_n$.

Let δ be a continuous linear operator on E such that $\delta(E_n) \subset E_{n+1}$, for all $n \in \mathbb{N}$. Then δ is the infinitesimal generator of a semigroup $\{T_t : t \geq 0\}$ in E iff for all $f \in E$ there is an $n(f) \in \mathbb{N}$ such that $\delta^{n(f)}(f) = 0$.

Proof The sufficiency is obvious.

We now show the necessity. By Proposition C.2, since E is barrelled, the semigroup is locally equicontinuous. Then $\{T_t : 0 \leq t \leq 1\}$ is equicontinuous. By [64], Theorem 2.4, the set $c(f) = \{T_t f : 0 \leq t \leq 1\}$ is bounded for all $f \in E$. If we set $\tilde{E}_n = \bigoplus_{r=1}^n E_r$, then $E = \text{strict-}\lim_{\rightarrow} \tilde{E}_n$ and \tilde{E}_n is closed in \tilde{E}_{n+1} for all $n \in \mathbb{N}$. By Proposition A.35 of Appendix A there is an $m(f) \in \mathbb{N}$, such that $c(f) \subset \tilde{E}_{m(f)}$. We show by induction that $\delta^n(f) \in \tilde{E}_{m(f)}$ for all $n \geq 0$. First $\delta^0(f) = f \in E_{m(f)}$. Suppose now that for $0 \leq n \leq k$, $\delta^n(f) \in E_{m(f)}$. Then for $0 < h \leq 1$ we have

$$h^{-k-1}(k+1)! [T_h f - \sum_{\ell=0}^k (\ell!)^{-1} h^\ell \delta^\ell(f)] \in E_{m(f)}$$

Letting $h \rightarrow 0$, we get from Proposition C.8 and the fact that $E_{m(f)}$ is complete

$$\delta^{k+1}(f) \in E_{m(f)}$$

The assumption $\delta(E_n) \subset E_{n+1}$ for all $n \in \mathbb{N}$ and $\delta^n(f) \in E_{m(f)}$ for all $n \in \mathbb{N}$ imply that there is an $n(f) \in \mathbb{N}$ such that $\delta^{n(f)}(f) = 0$. ■

APPENDIX D

A FAMILY OF MODULAR AUTOMORPHISMS

As promised in Section 5.1 we give here a procedure to construct closed real subspaces having the properties specified in Proposition 5.1

Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be a complex separable infinite dimensional Hilbert space with orthonormal basis $\{e_n\}_{n \in \mathbb{N}}$. \mathcal{H} can also be viewed as a real Hilbert space when equipped with the inner product $(\cdot, \cdot) = \operatorname{Re} \langle \cdot, \cdot \rangle$.

On the set $H = \{\sum_{n \in \mathbb{N}} \alpha_n e_n : \alpha_n \in \mathbb{R}, \sum_{n \in \mathbb{N}} \alpha_n^2 < +\infty\}$ the inner products, $\langle \cdot, \cdot \rangle$ and (\cdot, \cdot) coincide and turns H into a real Hilbert space; it is easy to see that $H+iH = \mathcal{H}$. Let now, K and L be closed real subspaces of H such that $K \cap L = \{0\}$ and $K+L$ is dense in H ; then $\mathcal{K} = K+iL$ is a closed real subspace of \mathcal{H} because it is the sum of two orthogonal (with respect to (\cdot, \cdot)) closed real subspaces. Evidently $i\mathcal{K} = L+iK$ and $\mathcal{K} + i\mathcal{K} = (K+L)+i(K+L)$, which implies respectively, that

$$\mathcal{K} \cap i\mathcal{K} = \{0\} \text{ and } \mathcal{K} + i\mathcal{K} \text{ is dense in } H+iH = \mathcal{H}.$$

To illustrate the above method we discuss an example, based on some ideas given in [65], Section 6.5, Problem 2.

Example Let \mathcal{H} be a complex separable infinite dimensional Hilbert space with orthonormal basis $\{e_n\}_{n \in \mathbb{N}}$. Define now

$$K = \{\sum_{n \in \mathbb{N}} \alpha_n e_{2n} : \alpha_n \in \mathbb{R}, \sum_{n \in \mathbb{N}} \alpha_n^2 < +\infty\}$$

and

$$L_\lambda = \{\sum_{n \in \mathbb{N}} \beta_n b_n : b_n = (1+\lambda_n^2)^{-\frac{1}{2}}(e_{2n} + \lambda_n e_{2n+1}), \beta_n \in \mathbb{R}, \sum_{n \in \mathbb{N}} \beta_n^2 < +\infty\}$$

where $\lambda = \{\lambda_n\}_{n \in \mathbb{N}}$ is a sequence of non-zero real numbers that tends to zero. It is not difficult to show that K and L_λ are closed real subspaces of H such that $K \cap L_\lambda = \{0\}$ and $\overline{K+L_\lambda} = H$, where as above

$H = \{\sum_{n \in \mathbb{N}} \alpha_n e_n : \alpha_n \in \mathbb{R}, \sum_{n \in \mathbb{N}} \alpha_n^2 < +\infty\}$. Further, it can be shown that

a vector $\sum_{n+1 \in \mathbb{N}} (\gamma_n e_{2n} + \delta_n e_{2n+1})$ in H is in $K+L_\lambda$ iff $\sum_{n+1 \in \mathbb{N}} (\delta_n / \lambda_n)^2 < +\infty$.

Therefore $K+L_\lambda \neq H$ since $\lambda_n \rightarrow 0$. If $\mathcal{K}_\lambda = K+iL_\lambda$, then \mathcal{K}_λ is a closed real subspace of \mathcal{H} such that $\mathcal{K}_\lambda \cap i\mathcal{K}_\lambda = \{0\}$ and $\mathcal{K}_\lambda + i\mathcal{K}_\lambda$ is dense in \mathcal{H} . Let P_λ and Q_λ be the real orthogonal projections onto \mathcal{K}_λ and $i\mathcal{K}_\lambda$ respectively, and $R_\lambda = P_\lambda + Q_\lambda$; the eigenvalues of R_λ are $\mu_{\pm, n}^\lambda = 1 \pm (1 + \lambda_n^2)^{-\frac{1}{2}}$, $n+1 \in \mathbb{N}$, and the spectrum of R_λ is $\{0, 2\} \cup \{\mu_{\pm, n}^\lambda\}_{n+1 \in \mathbb{N}}$. The eigenvectors corresponding to the eigenvalues $\mu_{\pm, n}^\lambda$ are $\alpha_{\pm, n}^\lambda e_{2n} + \beta_{\pm, n}^\lambda e_{2n+1}$ where $\alpha_{\pm, n}^\lambda = (\mu_{\pm, n}^\lambda / 2)^{\frac{1}{2}}$ and $\beta_{\pm, n}^\lambda = \pm (\mu_{\mp, n}^\lambda / 2)^{\frac{1}{2}}$. The action of the modular group $\Delta_\lambda^{it} = (2 - R_\lambda)^{it} R_\lambda^{-it}$ on these vectors is given by

$$\Delta_\lambda^{it} (\alpha_{\pm, n}^\lambda e_{2n} + \beta_{\pm, n}^\lambda e_{2n+1}) = (\mu_{\mp, n}^\lambda / \mu_{\pm, n}^\lambda)^{it} (\alpha_{\pm, n}^\lambda e_{2n} + \beta_{\pm, n}^\lambda e_{2n+1})$$

which can then be extended to all of \mathcal{H} by linearity and continuity.

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